Quantum Coordination Rates in Multi-User Networks

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Abstract

Quantum coordination is considered in networks with classical and quantum links. We begin with networks with *classical links*, and characterize the generation of separable and classical-quantum correlations in three primary models: 1) a two-node network with limited common randomness (CR), 2) a no-communication network, and 3) a broadcast network, which consists of a single sender and two receivers. We establish the optimal tradeoff between the classical communication and CR rates in each setting, thus characterizing the minimal resources for simulating classical-quantum correlations.

Next, we consider coordination in networks with *quantum links*. We study the following models: 1) a cascade network with limited entanglement, 2) a broadcast network, and 3) a multiple-access network with two senders and a single receiver. We establish the optimal tradeoff between quantum communication and entanglement rates in each setting, characterizing the minimal resources for entanglement coordination. The examples demonstrate that coordination of entanglement and coordination of separable correlations behave differently. At last, we show the implications of our results on nonlocal games with quantum strategies.

Index Terms

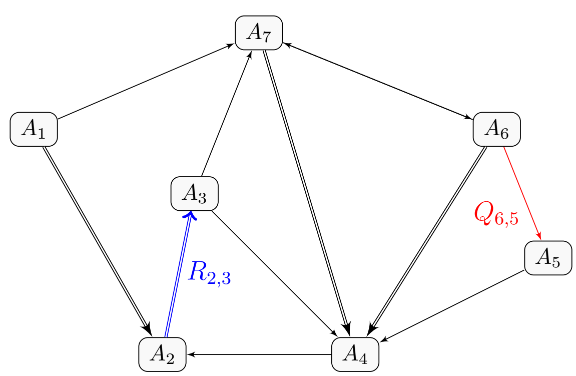
Quantum communication, coordination, reverse Shannon theorem, entanglement distribution.

I. Introduction

State distribution and coordination are important in quantum communication [3], computation [4], and cryptography [5]. The quantum coordination problem can be described as follows. Consider a network that consists of N nodes, where Node i can perform an encoding operation $\mathcal{E}_i$ on a quantum system A_i, and its state should be in a certain correlation with the rest of the network nodes. An example is shown in Figure 1. The objective is to *simulate* a specific joint state $\omega_{A_1,A_2,\ldots,A_N}$, i.e., a noisy correlation. Node i can send qubits to Node j via a quantum channel at a limited rate $Q_{i,j}$. The nodes may also share limited entanglement resources, prior to their communication. The optimal performance is characterized by the quantum communication rates $Q_{i,j}$ that are necessary and sufficient for simulating the desired quantum correlation. Alternatively, the nodes may send bits using classical communication links at a limited rate $R_{i,j}$. Instances of the network coordination problem include channel/source simulation [6–12], state merging [13–15], state redistribution [16–19], entanglement dilution [20–22], randomness extraction [23, 24], source coding [25–28], and many others.

Coordination can be viewed as a pre-processing step, i.e., preparation, prior to the execution of a communication task. In particular, noisy correlations are a valuable resource for information-theoretically secure cryptographic protocols which are otherwise impossible. Specifically, commitment primitives are widely used as a sub-protocol in applications [29, 30], such

Fig. 1. Coordination in a network that consists of $N = 7$ nodes. Some of the links are classical, while others are quantum. For instance, Node 2 sends classical bits to Node 3 at a limited rate $R_{2,3}$. Whereas, Node 6 sends qubits to Node 5 at a quantum rate $Q_{6,5}$.



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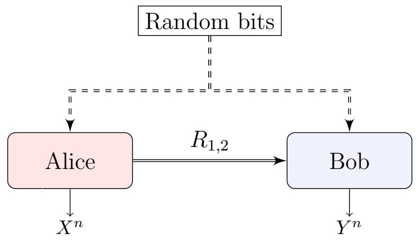


Fig. 2. Classical two-node network

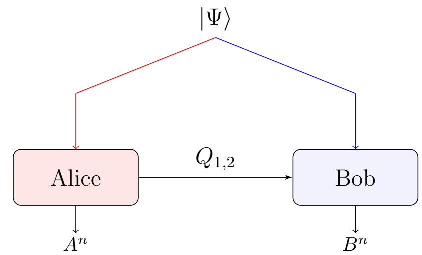


Fig. 3. Quantum two-node network

as secure multi-party computation [31–33], contract signing [34], and zero-knowledge proofs [35–37]. In the commitment primitive, Alice commits to a specific value in a way that keeps the value hidden from Bob, until a later point when Alice chooses to reveal the committed value [38]. Metaphorically, Alice secures a message in a locked box and sends it to Bob, concealing its contents. She later provides Bob with the key to unlock the box and verify the committed value. The two critical properties of commitment are hiding, which keeps the committed value hidden from Bob, and binding, which prevents Alice from altering it after committing. Without computational assumptions and assuming noiseless communications, commitment is impossible [39–41]. The simulation of a noisy channel and coordination of imperfect correlations is thus instrumental for cryptography, as it facilitates information-theoretically secure protocols for cryptographic primitives [42].

Two-node classical coordination: In classical coordination, the goal is to simulate a joint probability distribution. In the basic two-node network, see Figure 2, two users would like to simulate a joint distribution p_{XY}. This can be achieved if and only if the *classical* communication rate $R_{1,2}$ is above Wyner's common information [43], defined as:

$$C(X;Y) \triangleq \min I(U;XY)\,, \tag{1}$$

where the minimum is taken over all auxiliary variables U that satisfy the Markov relation $X \multimap U \multimap Y$, and $I(U;XY)$ is the mutual information between U and (X,Y). One may also consider the case where the nodes share classical correlation resources, a priori, in the form of common randomness (CR). Given a sufficient amount of pre-shared CR, the desired distribution can be simulated if and only if $R_{1,2} \geq I(X;Y)$ [44, 45].

Two-node quantum coordination: In the quantum setting, see Figure 3, the goal is to simulate a joint state. A bipartite state ω_{AB} can be simulated if and only if the quantum communication rate is above the von Neumann entropy [46], i.e., $Q_{1,2} \geq H(\omega_B)$, with $H(\rho) = -\mathrm{Tr}(\rho \log(\rho))$, where ω_B is the reduced state of ω_{AB}. As opposed to the classical case, the optimal strategy for Alice is lossless compression of her source [47]. Now, suppose that the nodes share entanglement resources, prior to their communication. Based on the quantum reverse Shannon theorem [48], given sufficient entanglement, the desired state can be simulated if and only if the quantum communication rate satisfies $Q_{1,2} \geq \frac{1}{2} I(A;B)_\omega$, where $I(A;B)_\omega$ is the quantum mutual information.

II. Unified Framework

Coordination can be viewed as a unified framework for various models. We list a few closely-related problems: (see Table I)

1) **Schumacher compression:** In lossless compression, Alice encodes a source $\omega_B^{\otimes n}$ and sends $nQ_{1,2}$ qubits to Bob. As Bob recovers the source, they effectively simulate a purification $|\omega_{RB}\rangle$. Achievability requires $Q_{1,2} \geq H(B)_\omega$ [47].
2) **Channel resolvability:** Resolvability aims to approximate the output of a classical-quantum (c-q) channel using a uniformly distributed codebook [55]. The task is equivalent to the simulation of a c-q state ω_{XB} in a two-node network with a classical link. Channel resolvability can be achieved at a communication rate $R_{1,2} \geq I(X;B)_\rho$ [49, 55]. Resolvability is also referred to as c-q soft covering [56]. Recently, Atif et al. [57] have also introduced fully quantum soft covering.
3) **Entanglement embezzlement:** Van Dam and Hayden [50] showed that without any type of communication, any pure bipartite entangled state $|\omega_{AB}\rangle$ can be simulated with "embezzlement" from a finite catalyst [58], i.e., while removing a small amount from the catalyst. Specifically, they use catalyst states $|\mu\rangle \propto \sum \frac{1}{\sqrt{j}} |j\rangle_A \otimes |j\rangle_B$ of Schmidt rank n, and show that for embezzling with a fidelity $1-\varepsilon$ can be achieved with $n > m^{\frac{1}{\varepsilon}}$, where m is the Schmidt rank of $|\omega_{AB}\rangle$.
4) **Entanglement dilution:** Suppose that Alice and Bob would like to prepare a joint state $|\omega_{AB}\rangle$ using local opreations and classical communication (LOCC), and a pre-shared maximally entangled state of dimension $2^{nE_{1,2}}$. The simulation requires $E_{1,2} \geq H(B)_\omega$ and negligible classical communication [20, 21].
5) **Entanglement distillation:** Given n copies of a bipartite state, $\omega_{AB}^{\otimes n}$, Alice and Bob can distill entangled EPR pairs, $|\Phi_{AB}\rangle$, at an output rate that is bounded by the coherent information, $E_{1,2}^{\text{out}} \leq I(A\rangle B)_\omega$, using classical communication at a rate $R_{1,2} \geq H(A|B)_\omega$ (see [51]). The analysis relies on typical subspace measurements and measurements in the Fourier transformed basis [59–63].

TABLE I
QUANTUM PROTOCOLS

	Task	Type	Simulated state	Rates	Ref.
1	Schumacher compression	Coordination	$\vert\omega_{RB}\rangle$	$Q_{1,2} \geq H(B)_\omega$, $R_{1,2}=0$, $E_{1,2}=0$	[47]
2	Resolvability	Coordination	ω_{XB}	$Q_{1,2}=0$, $R_{1,2} \geq I(X;B)_\omega$, $E_{1,2}=0$	[49]
3	Entanglement embezzlement	Coordination	$\vert\mu\rangle \otimes \vert\omega_{AB}\rangle$	$Q_{1,2}=R_{1,2}=0$	[50]
4	Entanglement dilution	Coordination	$\vert\omega_{AB}\rangle$	$Q_{1,2}=0$, $R_{1,2}\approx 0$, $E_{1,2} \geq H(B)_\omega$	[21]
5	Entanglement distillation	Distillation	$\vert\Phi_{AB}\rangle$	$Q_{1,2}=0$, $R_{1,2} \geq H(A\vert B)_\omega$, $E^{\text{out}}_{1,2} \leq I(A\rangle B)_\omega$	[51]
6	Subspace transmission	Communication	$\omega_{RB} = (\text{id}\otimes\mathcal{N}_{A\to B})(\omega_{RA})$	$Q^{\text{out}}_{1,2} \leq I(R\rangle B)_\omega$, $E_{1,2}=0$	[52]
7	State merging	Coordination	ω_{AB}	$R_{1,2} \geq I(A;B)_\omega$, $Q_{1,2} \geq H(A\vert B)_\omega$	[15]
8	State splitting	Coordination	$\vert\omega_{ABR}\rangle$	$Q_{1,2} \geq \frac{1}{2}I(R;B)_\omega$, $E_{1,2} \geq \frac{1}{2}I(A;B)_\omega$	[53]
9	Father protocol	Communication	$\omega_{RBE} = (\text{id}\otimes\mathcal{U}^{\mathcal{N}}_{A\to BE})(\omega_{RA})$	$Q_{1,2} \leq \frac{1}{2}I(R;B)_\omega$, $E_{1,2} \geq \frac{1}{2}I(R;E)_\omega$	[44]
10	Mother protocol	Coordination	$\vert\Phi_{A''B''}\rangle \otimes \vert\omega_{ABR}\rangle$	$Q_{1,2} \geq \frac{1}{2}I(A;R)_\omega$, $E^{\text{out}}_{1,2} \leq \frac{1}{2}I(A;B)_\omega$	[54]
11	State redistribution	Coordination	$\vert\omega_{ABGR}\rangle$	$Q_{1,2} \geq \frac{1}{2}I(B;R\vert G)_\omega$, $Q_{1,2}+E_{1,2} \geq H(B\vert G)_\omega$	[16]
12	Channel simulation	Coordination	$\omega_{RBE} = (\text{id}\otimes\mathcal{U}^{\mathcal{N}}_{A\to BE})(\omega_{RA})$	$Q_{1,2} \geq \frac{1}{2}I(R;B)_\omega$, $E_{1,2} \geq \frac{1}{2}I(E;B)_\omega$	[48]

6) **Subspace transmission:** Consider the transmission of quantum information via a quantum channel $\mathcal{N}_{A\to B}$. Based on the Lloyd-Shor-Devetak Theorem [52, 64, 65], a qubit transmission rate $Q_{1,2}$ is achievable if $Q_{1,2} \leq I(R\rangle B)_\omega$, with respect to the output state $\omega_{RB} = (\text{id}\otimes\mathcal{N}_{A\to B})(\omega_{RA})$. This rate is not necessarily optimal in general.
7) **State merging:** Consider a mixed state ω_{AB}, shared between Alice and Bob. In the state merging protocol, Alice sends her part to Bob using classical and quantum communication, at rates $R_{1,2} \geq I(A;B)_\omega$ and $Q_{1,2} \geq H(A|B)_\omega$, respectively [14, 15]. If $H(A|B)_\omega < 0$, LOCC is sufficient and quantum communication is not required.
8) **State splitting:** This is the reverse task, where Alice holds both A and B, and would like to send B to Bob [66]. Let $|\omega_{ABR}\rangle$ be a purification. In the state splitting protocol, Alice and Bob use quantum communication and pre-shared entanglement, at rates $Q_{1,2} \geq \frac{1}{2}I(R;B)_\omega$ and $E_{1,2} \geq \frac{1}{2}I(A;B)_\omega$, respectively [53, 54, 67, 68].
9) **The father protocol:** Consider entanglement-assisted communication via a quantum channel $\mathcal{N}_{A\to B}$. Given unlimited entanglement assistance, a qubit transmission rate $Q_{1,2}$ is achievable if and only if $Q_{1,2} \leq \frac{1}{2}I(R;B)_\omega$ (see [44, 48]). The information transmission rate above can be achieved with an entanglement rate of $E_{1,2} \geq \frac{1}{2}I(R;E)_\phi$, for $\omega_{RBE} = (\text{id}\otimes\mathcal{U}^{\mathcal{N}}_{A\to BE})(\omega_{RA})$, where $\mathcal{U}^{\mathcal{N}}_{A\to BE}$ is a Stinespring dilation of the channel $\mathcal{N}_{A\to B}$.
10) **The mother protocol:** The protocol is known under different names, e.g., quantum state transfer, fully quantum Slepian–Wolf (FQSW) [54], and coherent state merging [19, 69]. The mother protocol is also related to the father protocol by source-channel duality [53]. In the course of the protocol, Alice and Bob perform state merging and also distill entanglement at a rate $E^{\text{out}}_{1,2} \leq \frac{1}{2}I(A;B)_\omega$ [53, 54]. The main tool can be viewed as a decoupling theorem.
11) **State redistribution:** Consider a joint state $|\omega_{ABGR}\rangle$, where Alice holds A and B, Bob has access to G, and R is a purifying reference system. Alice would like to send B to Bob [16–18]. The analysis is based on the decoupling approach as well. The state redistribution setting was also considered in the one-shot case by Berta et al. [19]. State redistribution generalizes several protocols, such as Schumacher's compression [47], state merging [14, 15] and splitting [54].
12) **Channel simulation:** According to the classical reverse Shannon theorem [44], a classical channel of capacity C can be simulated at a classical rate $R_{1,2}$ if and only if $R_{1,2} \geq C$, given sufficient common randomness [45]. The quantum analog is not necessarily true in general, yet it holds for a product input state, $\omega_{RA}^{\otimes n}$ [48]. In this case, achievability was shown for $Q_{1,2} \geq \frac{1}{2}I(R;B)_\omega$ and $E_{1,2} \geq \frac{1}{2}I(B;E)_\omega$, for $\omega_{RBE} = (\text{id}\otimes\mathcal{U}^{\mathcal{N}}_{A\to BE})(\omega_{RA})$. The analysis shows that given pre-shared entanglement embezzlement, channel simulation can be achieved without backward communication [48]. Given LOCC, the entanglement cost for simulation is related to the entanglement of formation [7].

Multi-user versions of the protocols above have also been studied extensively in recent years. Abeyesinghe et. al. [54] use the mother protocol to generate distributed compression protocols for correlated quantum sources. Other results on quantum distributed compression can be found in [70–73]. Simulation of broadcast and multiple-access channels is considered in [9, 74] and [75], respectively. George and Cheng [76] have recently studied multipartite state splitting. Multi-user distillation and manipulation is considered in [77–83]. Streltsov et al. [84] studied multipartite state merging. A father protocol for broadcast

TABLE II
MODELS AND RESULTS

C/Q links	Model	Figure	Theorem	Analysis
Classical	Two nodes	4	1	Sec. VIII
	No communication	5	4	Sec. IX
	Broadcast	6	5	Sec. X
Quantum	Cascade	7	6	Sec. XI
	Broadcast	8	8	Sec. XII
	Multiple-access	9	9	Sec. XIII

and multiple-access channels is presented in [85] and [86], respectively.

This brief overview is by no means exhaustive and is only meant to provide some background in order to place this contribution in context.

III. END UNIFIED FRAMEWORK

Multi-node quantum coordination: In this work, we consider quantum coordination in networks with either classical links or quantum links. The models of coordination in multi-user networks with classical links are motivated by quantum-enhanced Internet of Things (IoT) networks in which the communication links are classical [87–90], and the study of coordination in models including quantum links is motivated by applications such as the quantum Internet and quantum repeaters [91]. In each network, we determine the optimal coordination rates, characterizing the minimal resources required in order to simulate a joint quantum state among multiple parties. We further discuss the implications of our results on nonlocal quantum games. In particular, coordination in the broadcast network in Figure 8 can be viewed as a sequential game, where a coordinator (the sender) provides the players (the receivers) with quantum resources. In the course of the game, the referee sends questions, X^n and Y^n, to each player, and they respond with B^n and C^n. In order to win the game with a certain probability, the communication rates must satisfy the constraints with respect to an appropriate correlation.

Our work is divided into two parts, focusing on classical links and quantum links. Table II summarizes the models and results considered in this work. The first column indicates the communication type, classical or quantum, the second reads the network model, and the third provides the respective figure. In addition, we give the respective theorem and the section of the analysis. The networks that are considered in this work are fundamental building blocks for general network coordination, as illustrated in Figure 1.

In the analysis, we use different techniques for different networks, including quantum resolvability results [55, 92, 93], random coding, the state redistribution theorem [17], and the Schumacher compression protocol. In the broadcast network with quantum links, we assume that Alice does not have prior correlation with Bob and Charlie's resources X^n and Y^n. Therefore, the standard techniques of state redistribution [17] or quantum source coding with side information [94] are not suitable for our purposes. Instead, we generalize the method of classical binning [95] to handle the quantum case.

A. Classical Links

We first consider quantum coordination in three multi-user networks with classical communication links, where Node i sends classical bits to Node j at a limited rate $R_{i,j}$. While classical links cannot generate entanglement, we may consider the simulation of classical-quantum (c-q) and separable states in multi-user networks, where shared random bits are available to the network users at a limited rate. This resource is referred to as common randomness (CR).

We study three networks with classical links. Our results are summarized below.

1) Two-Node Network: Consider a two-user network as in Figure 4. Alice and Bob aim to simulate a c-q state $\omega_{XB}^{\otimes n}$. Before communication begins, Alice and Bob share CR in a limited bit rate R_0. Then, Alice sends classical bits at a rate $R_{1,2}$ to Bob. We characterize the optimal tradeoff between the required rate of description and the amount of CR used. Specifically, a rate pair $(R_0, R_{1,2})$ is called achievable if there exists a sequence of coordination codes such that the encoded state $\rho_{X^nB^n}$ is ε_n-close to $\omega_{XB}^{\otimes n}$, where ε_n tends to zero as $n \to \infty$. We show that coordination can be achieved iff the rate pair $(R_0, R_{1,2})$ satisfies

$$R_{1,2} \geq I(X;U)_\sigma \,, \tag{2}$$

$$R_0 + R_{1,2} \geq I(XB;U)_\sigma \,, \tag{3}$$

for some c-c-q extension σ_{XUB} of the form

$$\sigma_{XUB} = \sum_{(x,u)\in\mathcal{X}\times\mathcal{U}} p_{XU}(x,u)\, |x\rangle\langle x|_X \otimes |u\rangle\langle u|_U \otimes \theta_B^u \,. \tag{4}$$

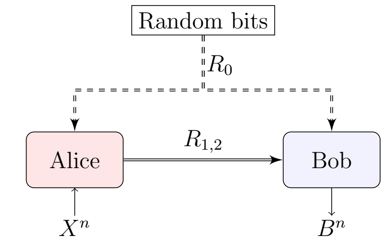


Fig. 4. Two-node network with classical links

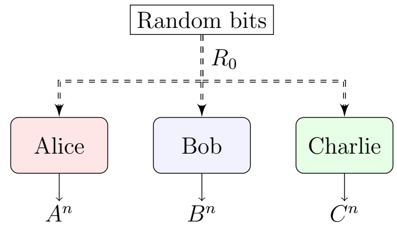


Fig. 5. No-communication network with common randomness

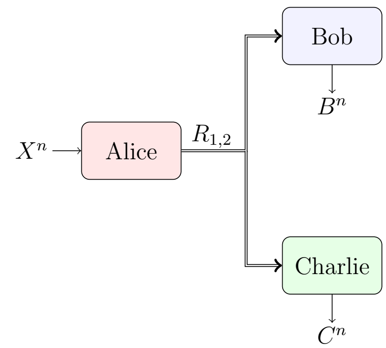


Fig. 6. Broadcast network with classical links

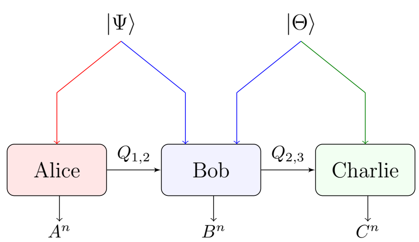


Fig. 7. Cascade network with quantum links and pre-shared entanglement

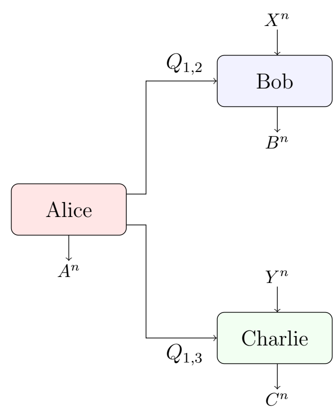


Fig. 8. Broadcast network with quantum links

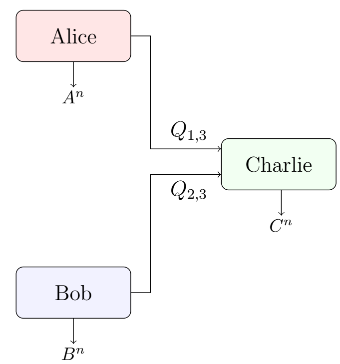


Fig. 9. Multiple-access network with quantum links

2) No-Communication Network: Our second model is a no-communication network, see Figure 5, where three users, Alice, Bob, and Charlie would like to simulate a separable state ω_{ABC}, given CR at a rate R_0, and no communication. We show that the optimal CR rate is $R_0 = \inf I(U;ABC)_\sigma$, where the infimum is over the set of all extensions

$$\sigma_{UABC} = \sum_{u \in \mathcal{U}} p_U(u) \ |u\rangle\langle u|_U \otimes \theta_A^u \otimes \theta_B^u \otimes \theta_C^u \tag{5}$$

such that $\sigma_{ABC} = \omega_{ABC}$. Note that A, B and C are uncorrelated when conditioned on U. The no-communication network was independently considered by George et al. [10] (see also [96, 97]).

3) Broadcast Network: In the broadcast network in Figure 6, a single sender and two receivers wish to simulate a classical-quantum-quantum (c-q-q) state ω_{XBC}. We establish that the state ω_{XBC} can be simulated in the broadcast network in Figure 6 iff the rate pair $(R_0, R_{1,2})$ satisfies

$$R_{1,2} \geq I(X;U)_\sigma \,, \tag{6}$$
$$R_0 + R_{1,2} \geq I(XBC;U)_\sigma \,, \tag{7}$$

for an extension σ_{XUBC} that satisfies a Markov property. Recently, George and Cheng [76] considered a similar setting of state splitting.

B. Quantum Links

In the second part of our work, we consider coordination with quantum links, where Node i sends qubits to Node j at a limited rate $Q_{i,j}$. We study three multi-user networks of this form.

1) Cascade Network: We begin with the cascade network in Figure 7. Alice, Bob, and Charlie wish to simulate a joint quantum state ω_{ABC}. Let $|\omega_{ABCR}\rangle$ be a purification of the desired state. Before communication begins, each party shares entanglement with their nearest neighbor, at a limited rate.

Now, Alice sends qubits to Bob at a rate $Q_{1,2}$, and thereafter, Bob sends qubits to Charlie at a rate $Q_{2,3}$. We show that ω_{ABC} can be simulated iff the rate tuple $(Q_{1,2}, E_{1,2}, Q_{2,3}, E_{2,3})$ satisfies

$$Q_{1,2} \geq \frac{1}{2} I(BC;R)_\omega \,, \tag{8}$$

$$Q_{1,2} + E_{1,2} \geq H(BC)_\omega \,, \tag{9}$$

$$Q_{2,3} \geq \frac{1}{2} I(C;RA)_\omega \,, \tag{10}$$

$$Q_{2,3} + E_{2,3} \geq H(C)_\omega \,, \tag{11}$$

where $E_{i,j}$ is the entanglement rate between Node i and Node j and $|\omega_{ABCR}\rangle$ is a purification of ω_{ABC}.

We provide two examples showing how the capacity behavior changes when simulating a mixture versus a tripartite entangled state (see Figure 14).

2) Quantum Broadcast Network: Next, we study the quantum broadcast network shown in Figure 8. Consider a network with a single sender, Alice, and two receivers, Bob and Charlie, where the latter are provided with classical sequences of information X^n and Y^n. We show that the state ω_{XYABC} can be simulated iff the rate pair $(Q_{1,2}, Q_{1,3})$ satisfies:

$$Q_{1,2} \geq H(B|X)_\omega \,, \tag{12}$$

$$Q_{1,3} \geq H(C|Y)_\omega , \tag{13}$$

where ω_{XYABC} is the desired joint state.

3) Multiple-Access Network: The third quantum-link setting is the multiple-access network shown in Figure 9. In this setting we have two transmitters, Alice and Bob, and one receiver, Charlie. We observe that since there is no cooperation between the transmitters, a joint state ω_{ABC} can only be simulated if it is isometrically equivalent to a state of the form $\omega_{AC_1} \otimes \omega_{BC_2}$. We show that the state ω_{ABC} can be simulated iff the rate pair $(Q_{1,3}, Q_{2,3})$ satisfies:

$$Q_{1,3} \geq H(C_1)_\omega \,, \tag{14}$$

$$Q_{2,3} \geq H(C_2)_\omega \tag{15}$$

We further discuss the implications of our results on nonlocal quantum games. In particular, coordination in the broadcast network in Figure 8 can be viewed as a sequential game, where a coordinator (the sender) provides the players (the receivers) with quantum resources. In the course of the game, the referee sends questions, X^n and Y^n, to each player, and they respond with B^n and C^n. In order to win the game with a certain probability, the communication rates must satisfy the constraints with respect to an appropriate correlation.

The paper is organized as follows. In Section IV, we introduce the notation conventions. In Section V, we consider coordination of c-q and separable correlations in networks that consist of classical links. We present the coding definitions and results for the two-node network, the no-communication network, and the broadcast network in Subsections V-A, V-B, and V-C, respectively. In Section VI, we consider entanglement coordination in networks that consist of quantum links. We address the cascade, broadcast, and multiple-access networks, in Subsections VI-A, VI-B, and VI-C, respectively. In Section VII, we discuss the implications of our results on quantum nonlocal games. The analysis for the three networks with classical links is given in Sections VIII, IX, X, and for the three networks with quantum links in Sections XI, XII, XIII. Section XIV is dedicated to summary and discussion.

IV. Notation

We use standard notation in quantum information theory, as in [69], $X, Y, Z, \ldots$ are discrete random variables on finite alphabets $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, ..., respectively, The distribution of X is specified by a probability mass function (pmf) $p_X(x)$ on $\mathcal{X}$. The set of all pmfs over $\mathcal{X}$ is denoted by $\mathcal{P}(\mathcal{X})$. We use $x^n = (x_i)_{i\in[n]}$ for a sequence in $\mathcal{X}^n$. A quantum state is described by a density operator, ρ_A, on the Hilbert space $\mathcal{H}_A$. Denote the set of all such operators by $\Delta(\mathcal{H}_A)$. A c-q channel is a map $\mathcal{N}_{X\to B} : \mathcal{X} \to \Delta(\mathcal{H}_B)$. A measurement is specified by a collection of operators $\{D_j\}$ that forms a positive operator-valued measure (POVM), i.e., $D_j \geq 0$ and $\sum_j D_j = \mathbb{1}$, where $\mathbb{1}$ is the identity operator. Given a bipartite state ρ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$, the quantum mutual information is defined as $I(A;B)_\rho = H(\rho_A) + H(\rho_B) - H(\rho_{AB})$, where $H(\rho) \equiv -\mathrm{Tr}[\rho \log(\rho)]$ is the von Neumann entropy. The conditional quantum entropy is defined as $H(A|B)_\rho = H(\rho_{AB}) - H(\rho_B)$, and the conditional quantum mutual information as $I(A;B|C)_\rho = H(A|C)_\rho + H(B|C)_\rho - H(A,B|C)_\rho$.

V. Classical links — Model Definitions and Results

We begin with networks with classical links. We consider three coordination settings with classical communication links as described below.

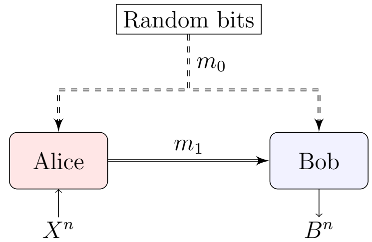


Fig. 10. Two-node network. Alice and Bob are provided with a common randomness element m_0, before the protocol begins. A classical sequence X^n is generated by Nature, and given to Alice. She then sends a message m_1 to Bob through a noiseless link. As Bob receives the message, he prepares the state of his output, B^n. The objective is to simulate n copies of a desired c-q state ω_{XB}, i.e., for the encoded state $\widehat{\rho}_{X^nB^n}$ to be arbitrarily close to $\omega_{XB}^{\otimes n}$.

A. Two-Node Network

Consider the two-node network in Figure 10. Here, we use simpler notation, $R_1 \equiv R_{1,2}$, for convenience. Alice and Bob wish to simulate a c-q state $\omega_{XB}^{\otimes n}$, using the following scheme. Node 1 (Alice) receives a classical source sequence x^n, drawn by Nature according to a given PMF p_X. The source sequence is encoded into an index m_1 at a rate R_1. Node 2 (Bob) is quantum. Both nodes have access to a CR element m_0 at a given rate R_0, i.e., m_0 is uniformly distributed over $\left[2^{nR_0}\right]$, and it is independent of X^n.

Formally, a $\left(2^{nR_0}, 2^{nR_1}, n\right)$ coordination code for the simulation of a c-q state ω_{XB} consists of a classical encoding channel, $F : \mathcal{X}^n \times [2^{nR_0}] \to [2^{nR_1}]$, and a c-q decoding channel $\mathcal{D}_{M_0M_1\to B^n}$. The protocol works as follows. A classical sequence $x^n \sim p_X^n$ is generated by Nature. Given the sequence x^n and the CR element m_0, Alice selects a random index,

$$m_1 \sim F(\cdot|x^n, m_0) \tag{16}$$

and sends it through a noiseless link. As Bob receives the message m_1 and the CR element m_0, he prepares the state

$$\rho_{B^n}^{(m_0,m_1)} = \mathcal{D}_{M_0M_1\to B^n}(m_0, m_1)\,. \tag{17}$$

Hence, the resulting joint state is

$$\widehat{\rho}_{X^nB^n} = \frac{1}{2^{nR_0}} \sum_{m_0\in[2^{nR_0}]} \sum_{x^n\in\mathcal{X}^n} \left(p_X^n(x^n)\, |x^n\rangle\!\langle x^n|_{X^n} \otimes \sum_{m_1\in[2^{nR_1}]} F(m_1|x^n, m_0) \rho_{B^n}^{(m_0,m_1)} \right). \tag{18}$$

Definition 1. A coordination rate pair (R_0, R_1) is achievable for the simulation of ω_{XB}, if for every $\varepsilon, \delta > 0$ and a sufficiently large n, there exists a $\left(2^{n(R_0+\delta)}, 2^{n(R_1+\delta)}, n\right)$ code that achieves

$$\left\|\widehat{\rho}_{X^nB^n} - \omega_{XB}^{\otimes n}\right\|_1 \le \varepsilon\,. \tag{19}$$

The coordination capacity region of the two-node network, $\mathcal{R}_{\text{2-node}}(\omega)$, with respect to the c-q state ω_{XB}, is the closure of the set of all achievable rate pairs.

The coordination capacity, $\mathsf{C}_{\text{2-node}}^{(0)}(\omega)$, without CR, is the supremum of rates R_1 such that $(0, R_1) \in \mathcal{R}_{\text{2-node}}(\omega)$. The CR-assisted coordination capacity, $\mathsf{C}_{\text{2-node}}^{(\infty)}(\omega)$, i.e., with unlimited CR, is the supremum of rates R_1 such that $(R_0, R_1) \in \mathcal{R}_{\text{2-node}}(\omega)$ for some $R_1 \ge 0$.

The optimal coordination rates for the two-node network are established below. Consider a given c-q state ω_{XB} that we wish to simulate. We now state our main result. Define the following set of c-c-q states. Let $\mathscr{S}_{\text{2-node}}(\omega)$ be the set of all c-c-q states

$$\sigma_{XUB} = \sum_{\substack{(x,u)\in\\ \mathcal{X}\times\mathcal{U}}} p_{XU}(x,u)\, |x\rangle\!\langle x|_X \otimes |u\rangle\!\langle u|_U \otimes \theta_B^u \tag{20a}$$

such that

$$\sigma_{XB} = \omega_{XB} \tag{20b}$$

for $|\mathcal{U}| \le |\mathcal{X}|^2[\dim(\mathcal{H}_B)]^2 + 1$. Notice that given a classical value $U = u$, there is no correlation between X and B.

Theorem 1. The coordination capacity region for the two-node network described in Figure 10 is given by the set

$$\mathcal{R}_{\text{2-node}}(\omega) = \bigcup_{\mathscr{S}_{\text{2-node}}(\omega)} \left\{ \begin{array}{rl} (R_0, R_1) \in \mathbb{R}^2 : \quad R_1 & \ge I(X;U)_\sigma\,, \\ R_0 + R_1 & \ge I(XB;U)_\sigma \end{array} \right\}. \tag{21}$$

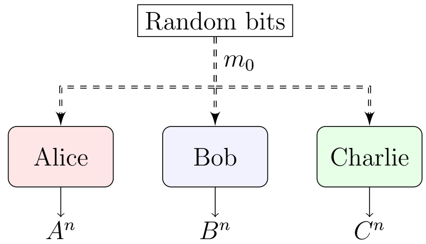


Fig. 11. No-communication network. Alice, Bob, and Charlie are only provided with a common randomness element m_0, and cannot communicate with one another. Each encoder applies a local encoding map to encode their output, A^n, B^n, and C^n, respectively. The objective is to simulate n copies of a desired separable state ω_{ABC}, i.e., for the encoded state $\widehat{\rho}_{A^nB^nC^n}$ to be arbitrarily close to $\omega_{ABC}^{\otimes n}$.

The proof for Theorem 1 is given in Section VIII. The following corollaries immediately follow.

Corollary 2 (Quantum Common Information [10]). The coordination capacity without CR is

$$\mathsf{R}^{(0)}_{\text{2-node}}(\omega) = \min_{\sigma_{XUB}\in\mathscr{S}_{\text{2-node}}(\omega)} I(XB;U)_\sigma \,. \tag{22}$$

Corollary 3. The CR-assisted coordination capacity, i.e., with unlimited CR, is given by

$$\mathsf{R}^{(\infty)}_{\text{2-node}}(\omega) \triangleq \min_{\sigma_{XUB}\in\mathscr{S}_{\text{2-node}}(\omega)} I(X;U)_\sigma \tag{23}$$

We note that in order to achieve the CR-assisted capacity, a CR rate of $R_0 = I(U;B|X)_\sigma$ is sufficient. If $B \equiv Y$ is classical, then we may substitute $U = Y$, which yields the capacity $\mathsf{R}^{(\infty)}_{\text{2-node}}(\omega) = I(X;Y)$, and it can be achieved with CR at rate $R_0 = H(Y|X)$ [45].

B. No-Communication Network

Consider a network that consists of three users: Alice, Bob and Charlie, holding quantum systems A, B, and C, respectively. The users cannot communicate, but they share a CR element m_0 at a rate R_0, as illustrated in Figure 11. Given m_0, each user prepares a quantum state separately.

A $\left(2^{nR_0}, n\right)$ coordination code for the no-communication network consists of a CR set $[2^{nR_0}]$, and three c-q encoding channels, $\mathcal{T}^{(1)}_{M_0\to A^n}$, $\mathcal{T}^{(2)}_{M_0\to B^n}$, and $\mathcal{T}^{(3)}_{M_0\to C^n}$. As Alice, Bob, and Charlie receive a realization j of the CR element, each uses their encoding map to prepare their respective state. prepares a quantum state, $\rho^j_{A^n} = \mathcal{T}^{(1)}_{M_0\to A^n}(m_0)$, $\rho^j_{B^n} = \mathcal{T}^{(2)}_{M_0\to A^n}(m_0)$, and $\rho^j_{C^n} = \mathcal{T}^{(3)}_{M_0\to C^n}(m_0)$, respectively. Hence,

$$\widehat{\rho}_{A^nB^nC^n} = \frac{1}{2^{nR_0}} \sum_{m_0\in[2^{nR_0}]} \mathcal{T}^{(1)}(m_0)\otimes\mathcal{T}^{(2)}(m_0)\otimes\mathcal{T}^{(3)}(m_0) \,. \tag{24}$$

Definition 2. A CR rate R_0 is achievable for the simulation of ω_{ABC}, if for every $\varepsilon, \delta > 0$ and a sufficiently large n, there exists a $\left(2^{n(R_0+\delta)}, n\right)$ coordination code that achieves

$$\left\|\widehat{\rho}_{A^nB^nC^n} - \omega_{ABC}^{\otimes n}\right\|_1 \le \varepsilon \,. \tag{25}$$

The coordination capacity $\mathsf{C}_{\text{NC}}(\omega)$, for the no-communication network, is the infimum of achievable rates R_0. If there are no achievable rates, we set $\mathsf{C}_{\text{NC}}(\omega) = +\infty$.

The optimal coordination rates for the no-communication network are established below. Consider a given quantum state ω_{ABC} that we wish to simulate. We now state our main result. Define the following set of state extensions. Let $\mathscr{S}_{\text{NC}}(\omega)$ be the set of all c-q-q-q states

$$\sigma_{UABC} = \sum_{u\in\mathcal{U}} p_U(u)\ |u\rangle\langle u|_U \otimes \theta^u_A \otimes \theta^u_B \otimes \theta^u_C \tag{26a}$$

such that

$$\sigma_{ABC} = \omega_{ABC} \,. \tag{26b}$$

Notice that given $U = u$, there is no correlation between A, B and C.

Theorem 4. The coordination capacity for the no-communication network described in Figure 11 is

$$\mathsf{C}_{\text{NC}}(\omega) = \inf_{\sigma_{UABC}\in\mathscr{S}_{\text{NC}}(\omega)} I(U;ABC)_\sigma \tag{27}$$

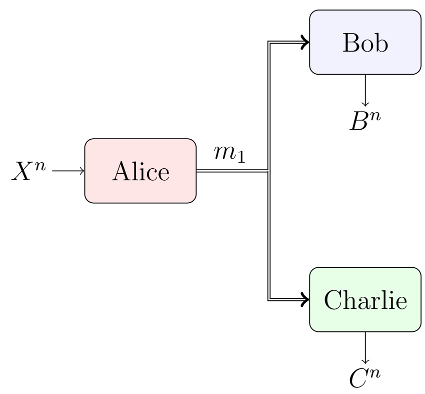


Fig. 12. Broadcast Network. Alice, Bob, and Charlie are provided with a common randomness element m_0, before the protocol begins. A classical sequence X^n is generated by Nature, and given to Alice. She then sends a message m_1 to Bob and Charlie. As Bob and Charlie receive the message, they encode the state of their respective output, B^n and C^n. The objective is to simulate n copies of a desired c-q-q state ω_{XBC}, i.e., for the encoded state $\widehat{\rho}_{X^n B^n C^n}$ to be arbitrarily close to $\omega_{XBC}^{\otimes n}$. The CR element is omitted from the figure for simplicity.

with the convention that an infimum over an empty set is $+\infty$.

The proof for Theorem 4 is given in Section IX. The no-communication network was independently considered by George et al. [10, Sec. VII].

Remark 1. Since the CR is classical, it cannot be used in order to create entanglement. Therefore, as Alice, Bob, and Charlie do not cooperate with one another, it is impossible to simulate entanglement. That is, we can only simulate separable states.

Remark 2. For a product state $\omega_{ABC} = \omega_A \otimes \omega_B \otimes \omega_C$, we may take U to be null, hence $\mathsf{C}_{\text{NC}}(\omega) = 0$. That is, simulation does not require CR between the users. On the other hand, if ω_{AB} is entangled, then there is no U that can satisfy (26), thus $\mathsf{C}_{\text{NC}}(\omega) = +\infty$. For a classically correlated state, $\omega_{ABC} = \frac{1}{2}\left(|000\rangle\langle 000| + |111\rangle\langle 111|\right)$, we have $\mathsf{C}_{\text{NC}}(\omega) = 1$, as one bit of CR is required in order to simulate such correlation.

C. Broadcast Network

Consider the broadcast network in Figure 12. A sender, Alice, and two receivers, Bob and Charlie, wish to simulate a c-q-q state ω_{XBC}, using the following scheme. Alice receives a classical source sequence $x^n \in \mathcal{X}^n$ drawn by Nature, i.i.d. according to a given PMF p_X. Alice encodes the source sequence into an index m_1 at a rate R_1. The other two nodes, of Bob and Charlie, are quantum. The three nodes have access to a CR element m_0 at a rate R_0. Similarly, a $\left(2^{nR_0}, 2^{nR_1}, n\right)$ coordination code consists of a classical encoding channel, $F : \mathcal{X}^n \times [2^{nR_0}] \to [2^{nR_1}]$, and two c-q decoding channels, $\mathcal{D}^{(\ell)}_{M_0 M_1 \to B_\ell^n}$, for $\ell \in \{1, 2\}$. Given x^n and the CR element m_0, Alice generates $m_1 \sim F(\cdot|x^n, m_0)$, and sends it to both Bob and Charlie, who then apply their decoding map.

The coordination capacity region of the broadcast network, $\mathcal{R}_{\text{BC}}(\omega)$, with respect to the c-q-q state ω_{XBC}, is defined in a similar manner as in Definition 1. Consider a given c-q-q state ω_{XBC} that we wish to simulate. Define the following set of c-c-q-q states. Let $\mathscr{S}_{\text{2-BC}}(\omega)$ be the set of all c-c-q-q states

$$\sigma_{XUBC} = \sum_{\substack{(x,u)\in \\ \mathcal{X}\times\mathcal{U}}} p_{XU}(x,u)\, |x\rangle\langle x|_X \otimes |u\rangle\langle u|_U \otimes \theta_B^u \otimes \eta_C^u \tag{28}$$

such that

$$\sigma_{XBC} = \omega_{XBC}\,. \tag{29}$$

Note that X, B, and C are uncorrelated given $U = u$.

Theorem 5. The coordination capacity region of the broadcast network in Figure 12 is the set

$$\mathcal{R}_{\text{BC}}(\omega) = \bigcup_{\mathscr{S}_{\text{BC}}(\omega)} \left\{ (R_0, R_1) \in \mathbb{R}^2 : \begin{array}{rl} R_1 & \geq I(X;U)_\sigma\,, \\ R_0 + R_1 & \geq I(XBC;U)_\sigma \end{array} \right\}. \tag{30}$$

The proof for Theorem 5 is given in Section X. The following corollaries immediately follow.

Remark 3. Since Alice's encoding is classical, she cannot distribute entanglement. Therefore, as Bob and Charlie do not cooperate with one another, it is impossible to simulate entanglement between Bob and Charlie. That is, we can only simulate states such that ω_{BC} is separable, as in the no-communication model (see Remark 1).

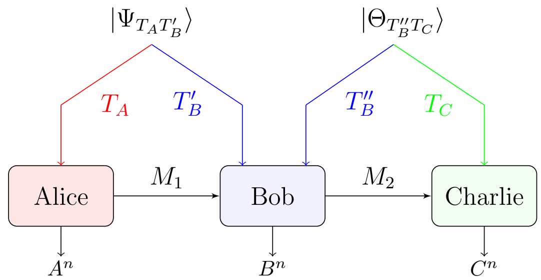


Fig. 13. Cascade network with rate-limited entanglement. Before communication begins, each party shares bipartite entanglement with their nearest neighbor. Alice prepares the state of her output A^n, as well as a "quantum description" M_1. She sends M_1 to Bob. As Bob receives M_1, he encodes the output B^n, along with his own quantum description, M_2. Next, Bob sends M_2 to Charlie. Upon receiving M_2, Charlie prepares the output state for C^n. The objective is to simulate n copies of a desired quantum state $|\omega_{RABC}\rangle$, i.e., for the encoded state $\widehat{\rho}_{R^n A^n B^n C^n}$ to be arbitrarily close to $|\omega_{RABC}\rangle^{\otimes n}$.

VI. Quantum links - Model Definitions and Results

We consider three coordination settings with quantum communication links, as described below. We then discuss the implications of the results obtained for the broadcast network shown in Subsection VI-B on nonlocal games.

A. Cascade network

Consider the cascade network with rate-limited entanglement, as depicted in Figure 13. In the Introduction section, we used the notation $Q_{i,j}$ for the communication rate from Node i to Node j. Here, we simplify the notation, and write $Q_1 \equiv Q_{1,2}$ and $Q_2 \equiv Q_{2,3}$, for convenience.

Alice, Bob, and Charlie would like to simulate a joint state $\omega_{ABC}^{\otimes n}$, where $\omega_{ABC} \in \Delta(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$. Let $|\omega_{RABC}\rangle$ be a purification of ω_{ABC}, where R can be viewed as Alice's reference. Before communication begins, each party shares bipartite entanglement with their nearest neighbor. The bipartite state $\left|\Psi_{T_A T_B'}\right\rangle$ indicates the entanglement resource shared between Alice and Bob, while $\left|\Theta_{T_B'' T_C}\right\rangle$ is shared between Bob and Charlie. The coordination protocol begins with Alice preparing the state of her output system A^n, as well as a "quantum description" M_1. She sends M_1 to Bob. As Bob receives M_1, he encodes the output B^n, along with his own quantum description, M_2. Next, Bob sends M_2 to Charlie. Upon receiving M_2, Charlie prepares the output state for C^n.

The transmissions M_1 and M_2 are limited to the quantum communication rates Q_1 and Q_2, while the pre-shared resources between Alice and Bob and between Bob and Charlie are limited to the entanglement rates E_1 and E_2, respectively.

Definition 3. A $(2^{nQ_1}, 2^{nQ_2}, 2^{nE_1}, 2^{nE_2}, n)$ coordination code for the cascade network in Figure 13 consists of:

- Two bipartite states $\left|\Psi_{T_A T_B'}\right\rangle$ and $\left|\Theta_{T_B'' T_C}\right\rangle$ on Hilbert spaces of dimension 2^{nE_1} and 2^{nE_2}, respectively, i.e.,

$$\dim(\mathcal{H}_{T_A}) = \dim(\mathcal{H}_{T_B'}) = 2^{nE_1}\,, \tag{31}$$
$$\dim(\mathcal{H}_{T_B''}) = \dim(\mathcal{H}_{T_C}) = 2^{nE_2}\,, \tag{32}$$

- two Hilbert spaces, $\mathcal{H}_{M_1}$ and $\mathcal{H}_{M_2}$, of dimension

$$\dim(\mathcal{H}_{M_j}) = 2^{nQ_j} \quad \text{for } j \in \{1,2\}\,, \tag{33}$$

  and
- three encoding maps,

$$\mathcal{E}_{\bar{A}^n T_A \to A^n M_1} : \Delta(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_{T_A}) \to \Delta(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_{M_1})\,, \tag{34}$$
$$\mathcal{F}_{M_1 T_B' T_B'' \to B^n M_2} : \Delta(\mathcal{H}_{M_1} \otimes \mathcal{H}_{T_B' T_B''}) \to \Delta(\mathcal{H}_B^{\otimes n} \otimes \mathcal{H}_{M_2})\,, \tag{35}$$

  and

$$\mathcal{D}_{M_2 T_C \to C^n} : \Delta(\mathcal{H}_{M_2} \otimes \mathcal{H}_{T_C}) \to \Delta(\mathcal{H}_C^{\otimes n})\,, \tag{36}$$

  corresponding to Alice, Bob, and Charlie, respectively.

The coordination protocol has limited communication rates Q_j and entanglement rates E_j, for $j \in \{1,2\}$. That is, before the protocol begins, Alice and Bob are provided with nE_1 entangled qubit pairs, while Bob and Charlie share nE_2 pairs. During the protocol, Alice transmits nQ_1 qubits to Bob, and then Bob transmits nQ_2 qubits to Charlie. See Figure 13. A detailed description of the protocol is given below.

The coordination protocol works as follows. Alice prepares the state $\omega_{R\bar{A}}^{\otimes n}$ locally, and applies the encoding map $\mathcal{E}_{\bar{A}^n T_A \to A^n M_1}$ on her share T_A of the entanglement resources. This results in the output state

$$\rho^{(1)}_{R^n A^n M_1 T'_B} = (\mathrm{id}_{R^n} \otimes \mathcal{E}_{\bar{A}^n T_A \to A^n M_1} \otimes \mathrm{id}_{T'_B})(\omega_{R\bar{A}}^{\otimes n} \otimes \Psi_{T_A T'_B}) . \tag{37}$$

She sends M_1 to Bob. Having received M_1, Bob uses it along with his shares $T'_B T''_B$ of the entanglement resources to encode, i.e., configure the state of B^n and M_2. To this end, he uses the map $\mathcal{F}_{M_1 T'_B T''_B \to B^n M_2}$, hence

$$\rho^{(2)}_{R^n A^n B^n M_2 T_C} = (\mathrm{id}_{R^n A^n} \otimes \mathcal{F}_{M_1 T'_B T''_B \to B^n M_2} \otimes \mathrm{id}_{T_C})(\rho^{(1)}_{R^n A^n M_1 T'_B} \otimes \Theta_{T''_B T_C}) . \tag{38}$$

Bob sends M_2 to Charlie, who applies the encoding channel $\mathcal{D}_{M_2 T_C \to C^n}$. This results in the final joint state,

$$\widehat{\rho}_{R^n A^n B^n C^n} = (\mathrm{id}_{R^n A^n B^n} \otimes \mathcal{D}_{M_2 T_C \to C^n}) \left(\rho^{(2)}_{R^n A^n B^n M_2 T_C}\right) . \tag{39}$$

The objective is that the final state $\widehat{\rho}_{R^n A^n B^n C^n}$ is arbitrarily close to the desired state $|\omega_{RABC}\rangle^{\otimes n}$.

Definition 4. A rate tuple (Q_1, Q_2, E_1, E_2) is achievable, if for every $\varepsilon, \delta > 0$ and a sufficiently large n, there exists a $(2^{n(Q_1+\delta)}, 2^{n(Q_2+\delta)}, 2^{n(E_1+\delta)}, 2^{n(E_2+\delta)}, n)$ coordination code satisfying

$$\left\|\widehat{\rho}_{R^n A^n B^n C^n} - \omega_{RABC}^{\otimes n}\right\|_1 \leq \varepsilon. \tag{40}$$

The coordination capacity region with respect to the state ω_{RABC} is defined as the closure of the set of all achievable rate tuples. We denote the coordination capacity region of the cascade network, with quantum links and rate-limited entanglement, by $\mathcal{Q}_{\text{Cascade}}(\omega)$.

Remark 4. Coordination in the cascade network can also be represented as as a resource inequality [98]

$$Q_1[q \to q]_{A \to B} + E_1[qq]_{AB} + Q_2[q \to q]_{B \to C} + E_2[qq]_{BC} \geq \langle \omega_{RABC} \rangle \tag{41}$$

where the resource units $[q \to q]$, $[qq]$, and $\langle \omega_{RABC} \rangle$ represent a single use of a noiseless qubit channel, an EPR pair, and the desired state ω_{RABC}, respectively.

The optimal coordination rates for the cascade network are established below.

Theorem 6. Consider a desired state $|\omega_{RABC}\rangle$. The coordination capacity region for the cascade network described in Figure 13 is given by the set

$$\mathcal{Q}_{\text{Cascade}}(\omega) = \left\{ \begin{array}{rrl} (Q_1, E_1, Q_2, E_2) : & Q_1 & \geq \frac{1}{2} I(BC; R)_\omega , \\ & Q_1 + E_1 & \geq H(BC)_\omega , \\ & Q_2 & \geq \frac{1}{2} I(C; RA)_\omega , \\ & Q_2 + E_2 & \geq H(C)_\omega \end{array} \right\} . \tag{42}$$

The proof for Theorem 6 is provided in Section XI.

Corollary 7. For a pure state $|\omega_{ABC}\rangle$, the coordination capacity region for the cascade network is given by the set

$$\mathcal{Q}_{\text{Cascade}}(\omega) = \left\{ \begin{array}{rrl} (Q_1, E_1, Q_2, E_2) : & Q_1 + E_1 & \geq H(BC)_\omega , \\ & Q_2 & \geq \frac{1}{2} I(C; A)_\omega , \\ & Q_2 + E_2 & \geq H(C)_\omega \end{array} \right\} . \tag{43}$$

The examples below demonstrate that coordination of entanglement and coordination of separable correlations behave differently.

Example 1 (Mixture). Let $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_C$ be Hilbert spaces of dimension 3, i.e., qutrits. Consider the simulation of a mixed state,

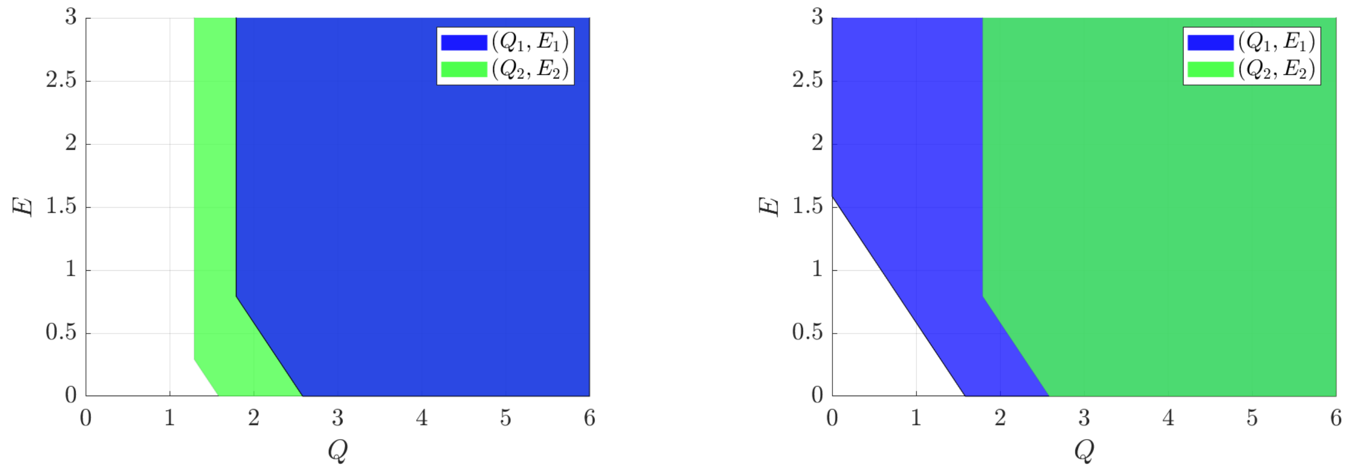
$$\omega_{ABC} = \frac{1}{6} \left(|012\rangle\langle 012| + |021\rangle\langle 021| + |102\rangle\langle 102| + |120\rangle\langle 120| + |201\rangle\langle 201| + |210\rangle\langle 210| \right) \tag{44}$$

The example is analogous to classical task assignment [99, Example 3]. The state above is thus purified by

$$|\omega_{RABC}\rangle = \frac{1}{6}\big(|0\rangle \otimes |012\rangle + |1\rangle \otimes |021\rangle + |2\rangle \otimes |102\rangle + |3\rangle \otimes |120\rangle + |4\rangle \otimes |201\rangle + |5\rangle \otimes |210\rangle \big) \tag{45}$$

where $\{|i\rangle\}_{i=0,\ldots,5.}$ forms an orthonormal basis for the reference system R. In this case, the coordination capacity region is given by

$$\mathcal{Q}_{\text{Cascade}}(\omega) = \left\{ \begin{array}{rrl} (Q_1, E_1, Q_2, E_2) : & Q_1 & \geq 1.7925, \\ & Q_1 + E_1 & \geq 2.5850, \\ & Q_2 & \geq 1.2925, \\ & Q_2 + E_2 & \geq 1.5850. \end{array} \right\} . \tag{46}$$



(a) Example 1: Mixture

(b) Example 2: Entanglement

Fig. 14. Achievable coordination regions in two examples. In the mixed-state example, Alice is required to send qubits to Bob at a *higher* rate than Bob to Charlie. In the entangled-state example, Alice is required to send qubits to Bob at a rate that is *lower* than Bob to Charlie. This occurs because of the "knowing less than nothing" phenomenon, i.e., the entropy of a subsystem is larger than the joint entropy.

The coordination capacity region $\mathcal{Q}_{\text{Cascade}}(\omega)$ is illustrated in Figure 14 (a), where the blue region shows the tradeoff between Alice's rates, Q_1 and E_1, and the green region is associated with Bob's rates, Q_2 and E_2.

Suppose that $E_1 = E_2$. As can be seen in the figure, Alice is required to send qubits to Bob at a higher rate than Bob to Charlie. This is intuitive since Alice encodes information for both Bob and Charlie, whereas Bob is only encoding Charlie's information.

Example 2 (Entanglement). Consider the simulation of a pure tripartite entangled state,

$$|\psi_{ABC}\rangle = \frac{1}{\sqrt{6}}\left(|012\rangle + |021\rangle + |102\rangle + |120\rangle + |201\rangle + |210\rangle\right) \tag{47}$$

According to Corollary 7, $|\psi_{ABC}\rangle^{\otimes n}$ can be simulated if and only if the rate tuple (Q_1, E_1, Q_2, E_2) belongs to the following set,

$$\mathcal{Q}_{\text{Cascade}}(\psi) = \left\{ \begin{array}{rrl} (Q_1, E_1, Q_2, E_2): & Q_1 + E_1 & \geq 1.5850\,, \\ & Q_2 & \geq 0.7925\,, \\ & Q_2 + E_2 & \geq 1.5850\,. \end{array} \right\}.$$

The coordination capacity region $\mathcal{Q}_{\text{Cascade}}(\psi)$ is illustrated in Figure 14 (b). As before, the blue region shows the tradeoff between Alice's rates, Q_1 and E_1, and the green region is associated with Bob's rates, Q_2 and E_2.

Suppose that $E_1 = E_2$. Here, as opposed to Example 1, Alice is required to send qubits to Bob at a rate that is *lower* than Bob to Charlie. This occurs because of the "knowing less than nothing" phenomenon [100]. That is, in the presence of entanglement, a subsystem can have a larger entropy compared to the joint system. The behavior in each example is completely different.

B. *Broadcast network*

Consider the broadcast network in Figure 15. This network, can be useful in analyzing refereed games and the required resources for achieving certain performances as described in section VII. As before, we simplify the notation $Q_{i,j}$ from the Introduction section, and write $Q_1 \equiv Q_{1,2}$ and $Q_2 \equiv Q_{1,3}$, for convenience. Consider a c-c-q-q-q state,

$$\omega_{XYABC} = \sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} p_{XY}(x,y)\, |x,y\rangle\langle x,y|_{X,Y} \otimes \left|\omega_{ABC}^{(x,y)}\right\rangle\!\left\langle\omega_{ABC}^{(x,y)}\right| \tag{48}$$

corresponding to a given ensemble of states $\left\{p_{XY}, \left|\omega_{ABC}^{(x,y)}\right\rangle\right\}$ in $\Delta(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$.

Alice, Bob, and Charlie would like to simulate ω_{XYABC}. Before communication takes place, the classical sequences X^n and Y^n are drawn from a common source $p_{XY}^{\otimes n}$. The sequence X^n is given to Bob, while Y^n is given to Charlie (see Figure 15).

Initially, Alice prepares the state of her output A^n, along with two quantum descriptions, M_1 and M_2. She then transmits M_1 and M_2, to Bob and Charlie, respectively, at limited qubit transmission rates, Q_1 and Q_2. As Bob receives the quantum description M_1, he uses it together with the classical sequence X^n to encode the output B^n, i.e., apply an encoding map to configure the output state. Similarly, Charlie receives M_2 and Y^n, and encodes his output C^n.

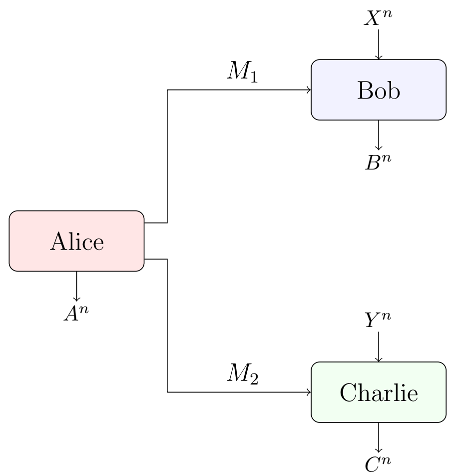


Fig. 15. Broadcast network. Before communication takes place, the classical sequences X^n and Y^n are drawn from a common source, and given to Bob and Charlie, respectively. Alice prepares the state of her output A^n, as well as her "quantum descriptions", M_1 and M_2. She transmits M_1 and M_2 to Bob and Charlie, respectively. As Bob receives M_1, he encodes the output B^n. Similarly, Charlie receives M_2, and encodes C^n. The objective is to simulate n copies of a desired quantum state ω_{ABC}, i.e., for the encoded state $\widehat{\rho}_{A^n B^n C^n}$ to be arbitrarily close to $\omega_{ABC}^{\otimes n}$.

Definition 5. A $(2^{nQ_1}, 2^{nQ_2}, n)$ coordination code for the broadcast network with side information described in Figure 15, consists of two Hilbert spaces, $\mathcal{H}_{M_1}$ and $\mathcal{H}_{M_2}$, of dimensions

$$\dim(\mathcal{H}_{M_j}) = 2^{nQ_j} \text{ for } j \in \{1,2\} \ , \tag{49}$$

and three encoding maps,

$$\mathcal{E}_{A^n \to A^n M_1 M_2} : \Delta(\mathcal{H}_A^{\otimes n}) \to \Delta(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_{M_1} \otimes \mathcal{H}_{M_2}), \tag{50}$$

$$\mathcal{F}_{X^n M_1 \to B^n} : \mathcal{X}^n \otimes \Delta(\mathcal{H}_{M_1}) \to \Delta(\mathcal{H}_B^{\otimes n}), \tag{51}$$

and

$$\mathcal{D}_{Y^n M_2 \to C^n} : \mathcal{Y}^n \otimes \Delta(\mathcal{H}_{M_2}) \to \Delta(\mathcal{H}_C^{\otimes n}). \tag{52}$$

corresponding to Alice, Bob, and Charlie, respectively. In the course of the protocol, Alice transmits nQ_1 qubits to Bob and nQ_2 qubits to Charlie, as illustrated in Figure 15.

Remark 5. In the quantum world, broadcasting a quantum state among multiple receivers is impossible by the no-cloning theorem. However, in the broadcast network in Figure 15, Alice sends two different "quantum messages" M_1 and M_2 to Bob and Charlie, respectively. Roughly speaking, Alice is broadcasting correlation. Since Alice prepares both quantum descriptions, M_1 and M_2, she can create correlation and generate tripartite entanglement between her, Bob, and Charlie.

The coordination protocol is described below. Alice applies her encoding map and prepares

$$\rho_{A^n M_1 M_2} = \mathcal{E}_{A^n \to A^n M_1 M_2}(\omega_A^{\otimes n}) \, . \tag{53}$$

She sends M_1 and M_2 to Bob and Charlie, respectively. Once Bob receives M_1 and the classical assistance, X^n, he applies his encoding map $\mathcal{F}_{X^n M_1 \to B^n}$. Similarly, Charlie receives M_2 and Y^n, and applies $\mathcal{D}_{Y^n M_2 \to C^n}$. Their encoding operations result in the following extended state:

$$\begin{aligned}\widehat{\rho}_{X^n Y^n A^n B^n C^n} = \sum_{x^n \in \mathcal{X}^n} \sum_{y^n \in \mathcal{Y}^n} p_{XY}^{\otimes n}(x^n, y^n) \, |x^n, y^n\rangle\langle x^n, y^n|_{X^n Y^n} \otimes \\ (\mathrm{id}_{A^n} \otimes \mathcal{F}_{X^n M_1 \to B^n} \otimes \mathcal{D}_{Y^n M_2 \to C^n}) \left(|x^n, y^n\rangle\langle x^n, y^n|_{\bar{X}^n \bar{Y}^n} \otimes \rho^{(1)}_{A^n M_1 M_2} \right) \, ,\end{aligned} \tag{54}$$

where $\bar{X}^n \bar{Y}^n$ are classical registers that store a copy of the (classical) sequences $X^n Y^n$, respectively. The goal is to encode such that the final state $\widehat{\rho}_{X^n Y^n A^n B^n C^n}$ is arbitrarily close to the desired state $\omega_{XYABC}^{\otimes n}$.

Definition 6. A rate pair (Q_1, Q_2) is achievable, if for every $\varepsilon, \delta > 0$ and a sufficiently large n, there exists a $(2^{n(Q_1+\delta)}, 2^{n(Q_2+\delta)}, n)$ coordination code satisfying

$$\left\| \widehat{\rho}_{X^n Y^n A^n B^n C^n} - \omega_{XYABC}^{\otimes n} \right\|_1 \le \varepsilon. \tag{55}$$

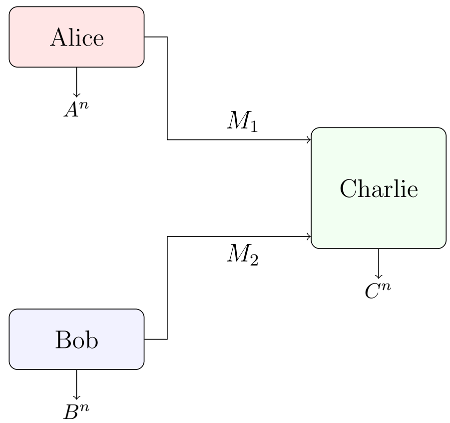


Fig. 16. Multiple-access network. Alice prepares the state of her output A^n, as well as her "quantum description", M_1. Similarly, Bob prepares the state of his output B^n, as well as M_2. They transmit M_1 and M_2 to Charlie. Upon receiving M_1 and M_2, Charlie encodes the output C^n. The objective is to simulate n copies of a desired quantum state ω_{ABC}, i.e., for the encoded state $\widehat{\rho}_{A^nB^nC^n}$ to be arbitrarily close to $\omega_{ABC}^{\otimes n}$.

The coordination capacity region of the broadcast network, $\mathcal{Q}_{\text{BC}}(\omega)$, with respect to the state ω_{XYABC}, is the closure of the set of all achievable rate pairs.

Remark 6. Notice that Alice has no access to X^n nor Y^n. Therefore, coordination can only be achieved for states ω_{XYABC} such that there is no correlation between A and XY, on their own. That is, the reduced state ω_{XYA} must have a product form,

$$\omega_{XYA} = \omega_{XY} \otimes \omega_A \,. \tag{56}$$

Since Alice does not share prior correlation with Bob and Charlie's resources X^n and Y^n, standard techniques, such as state redistribution [17] and quantum source coding with side information [94], are not suitable for our purposes. Instead, we introduce a quantum version of binning.

The optimal coordination rates for the broadcast network are established below.

Theorem 8. The coordination capacity region for the broadcast network described in Figure 15 is given by the set

$$\mathcal{Q}_{\text{BC}}(\omega) = \left\{ \begin{array}{ll} (Q_1, Q_2) \in \mathbb{R}^2 : & Q_1 \quad \geq H(B|X)_\omega \,, \\ & Q_2 \quad \geq H(C|Y)_\omega \end{array} \right\} . \tag{57}$$

The proof for Theorem 8 is provided in Section XII. The implications of this result on quantum nonlocal games are discussed in Section VII.

Remark 7. In the broadcast network, Alice, Bob, and Charlie can generate any form of entanglement. A simple example is a GHZ state, $|\omega_{ABC}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, taking X and Y to be null (say, $|\mathcal{X}| = |\mathcal{Y}| = 1$). In this case, coordination requires $Q_i \geq 1$.

C. Multiple-access network

Consider the multiple-access network in Figure 16. Alice, Bob, and Charlie would like to simulate a pure state $|\omega_{ABC}\rangle^{\otimes n}$, where $|\omega_{ABC}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. We simplify the notation and write $Q_1 \equiv Q_{1,3}$ and $Q_2 \equiv Q_{2,3}$. At first, Alice prepares the state of the quantum systems A^n and M_1, and Bob prepares the states of the quantum systems B^n and M_2. Alice and Bob send M_1 and M_2 to Charlie. Charlie then uses M_1 and M_2 to encode the system C^n. As in the previous settings, M_1 and M_2 are referred to as quantum descriptions, which are limited to the qubit transmission rates, Q_1 and Q_2, respectively.

Definition 7. A $(2^{\ell_1}, 2^{\ell_2}, n)$ coordination code for the multiple-access network described in Figure 16, consists of two Hilbert spaces, $\mathcal{H}_{M_1}$ and $\mathcal{H}_{M_2}$, of dimensions

$$\dim(\mathcal{H}_{M_j}) = 2^{\ell_j} \text{ for } j \in \{1,2\} \ , \tag{58}$$

and three encoding maps,

$$\mathcal{E}_{A^n \to A^n M_1} : \Delta(\mathcal{H}_A^{\otimes n}) \to \Delta(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_{M_1}) \,, \tag{59}$$

$$\mathcal{F}_{B^n \to B^n M_2} : \Delta(\mathcal{H}_B^{\otimes n}) \to \Delta(\mathcal{H}_B^{\otimes n} \otimes \mathcal{H}_{M_2}) \tag{60}$$

and

$$\mathcal{D}_{M_1 M_2 \to C^n} : \Delta(\mathcal{H}_{M_1} \otimes \mathcal{H}_{M_2}) \to \Delta(\mathcal{H}_C^{\otimes n}) \,, \tag{61}$$

corresponding Alice, Bob, and Charlie, respectively.

In the multiple-access network, Alice sends nQ_1 qubits to Charlie, while Bob sends nQ_2 qubits to Charlie. Specifically, Alice and Bob apply the encoding maps, preparing $\rho^{(1)}_{A^n M_1} \otimes \rho^{(2)}_{B^n M_2}$, where

$$\rho^{(1)}_{A^n M_1} = \mathcal{E}_{A^n \to A^n M_1}(\omega_A^{\otimes n}) \, , \; \rho^{(2)}_{B^n M_2} = \mathcal{F}_{B^n \to B^n M_2}(\omega_B^{\otimes n}) \, . \tag{62}$$

As Charlie receives M_1 and M_2, he applies his encoding map, which yields the final state,

$$\widehat{\rho}_{A^n B^n C^n} = (\mathrm{id}_{A^n B^n} \otimes \mathcal{D}_{M_1 M_2 \to C^n}) \, (\rho^{(1)}_{A^n M_1} \otimes \rho^{(2)}_{B^n M_2}) \, . \tag{63}$$

The ultimate goal of the coordination protocol is that the final state of $\widehat{\rho}_{A^n B^n C^n}$, is arbitrarily close to the desired state $\omega_{ABC}^{\otimes n}$.

Remark 8. Notice that since Charlie only acts on M_1 and M_2 which are encoded separately without coordination, we have $\widehat{\rho}_{A^n B^n} = \rho^{(1)}_{A^n} \otimes \rho^{(2)}_{B^n}$. Therefore, it is only possible to simulate states ω_{ABC} such that $\omega_{AB} = \omega_A \otimes \omega_B$. Since all purifications are isometrically equivalent [69, Theorem 5.1.1] there exists an isometry $V_{C \to C_1 C_2}$ such that

$$(\mathbb{1} \otimes V_{C \to C_1 C_2}) \, |\omega_{ABC}\rangle = |\phi_{AC_1}\rangle \otimes |\chi_{BC_2}\rangle \tag{64}$$

where $|\phi_{AC_1}\rangle$ and $|\chi_{BC_2}\rangle$ are purifications of ω_A and ω_B, respectively. If ω_{ABC} cannot be decomposed as in (64), then coordination is impossible in the multiple-access network.

Definition 8. A rate pair (Q_1, Q_2) is achievable, if for every $\varepsilon, \delta > 0$ and a sufficiently large n, there exists a $(2^{n(Q_1+\delta)}, 2^{n(Q_2+\delta)}, n)$ coordination code satisfying

$$\left\| \widehat{\rho}_{A^n B^n C^n} - \omega_{ABC}^{\otimes n} \right\|_1 \leq \varepsilon \, . \tag{65}$$

The coordination capacity region of the multiple-access network, $\mathcal{Q}_{\text{MAC}}(\omega)$, with respect to the state ω_{ABC}, is the closure of the set of all achievable rate pairs.

Remark 9. The resource inequality for coordination in the multiple-access network is

$$Q_1[q \to q]_{A \to C} + Q_2[q \to q]_{B \to C} \geq \langle \omega_{ABC} \rangle \tag{66}$$

(see resource definitions in Remark 4).

The optimal coordination rates for the multiple-access network are established below.

Theorem 9. Let $|\omega_{ABC}\rangle$ be a pure state as in (64). The coordination capacity region for the multiple-access network described in Figure 16 is given by the set

$$\mathcal{Q}_{\text{MAC}}(\omega) = \left\{ \begin{array}{lll} (Q_1, Q_2) \in \mathbb{R}^2 : & Q_1 & \geq H(A)_\omega \, , \\ & Q_2 & \geq H(B)_\omega \end{array} \right\} . \tag{67}$$

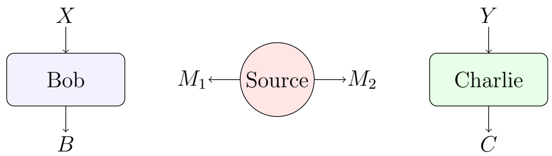
The proof for Theorem 9 is provided in Section XIII.

Remark 10. Consider the following trivial cases. For $|\omega_A\rangle \otimes |\omega_{BC}\rangle$, we have $Q_1 = 0$, as coordination does not require any communication from Alice to Charlie. Similarly, for $|\omega_{AC}\rangle \otimes |\omega_B\rangle$, we have $Q_2 = 0$. For example, if Alice and Charlie simulate a maximally entangled qubit state ω_{AC}, then the coordination region is $\mathcal{Q}_{\text{MAC}}(\omega) = \{(Q_1, 0) : Q_1 \geq 1\}$. Furthermore, if $|\omega_{ABC}\rangle = |\omega_{AB}\rangle \otimes |\omega_C\rangle$, then we also have $\omega_{AB} = |\omega_A\rangle \otimes |\omega_B\rangle$, hence coordination does not require communication at all.

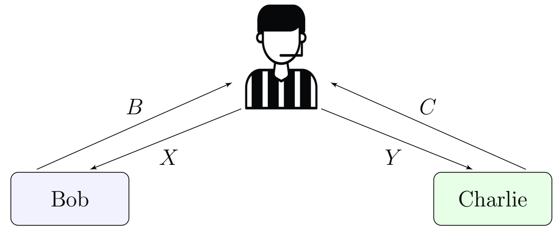
Remark 11. Consider a product of two maximally entangled qubit pairs, $|\omega_{ABC_1C_2}\rangle = |\phi_{AC_1}\rangle \otimes |\chi_{BC_2}\rangle$, where $\mathcal{H}_C \equiv \mathcal{H}_{C_1} \otimes \mathcal{H}_{C_2}$. The coordination capacity region is then $\mathcal{Q}_{\text{MAC}}(\omega) = \{(Q_1, Q_2) : Q_i \geq 1\}$. Now, suppose that Charlie performs a local Bell measurement on his qubits, C_1 and C_2, the entanglement is swapped such that A and B become maximally entangled. We will discuss the implications for the application of quantum repeaters in Subsection XIV-D.

VII. Nonlocal Games

In this section, we discuss the connection between quantum coordination and nonlocal games, focusing on the broadcast network. We begin with a brief review on refereed games in Subsection VII-A. We explain how coordination is useful for a sequential game in Subsection VII-D. We demonstrate the implications for the CHSH game in Subsection VII-E.



(a) Bell experiment setup.



(b) Refereed game setting.

Fig. 17. Bell experiments and refereed games. Figure (a) describes a Bell experiment setup, consisting of a source, and two observers. The source distributes physical systems M_1 and M_2 to Bob and Charlie respectively. Bob and Charlie choose to perform measurements X and Y each on his system, yielding classical results B and C respectively. Figure (b) describes a refereed game where a referee plays against both Bob and Charlie. The referee sends his questions X and Y to Bob and Charlie, they respond with answers B and C respectively. The round is won if the realization of the tuple $(X, Y, B, C) = (x, y, b, c)$ satisfies a specific condition set by the game rules.

A. Nonlocal Correlations

Nonlocal games are closely related to Bell experiments and quantum correlations [101]. In the typical setting for a Bell experiment, a source distributes two physical systems, M_1 and M_2, to two distant users. See Figure 17 (a). Here, we refer to the users as Bob (B) and Charlie (C). Upon receiving M_1 and M_2, each chooses to perform a measurement from a certain set of measurements. Denote the measurements chosen by Bob and Charlie by X and Y, respectively. The measurements yield the respective outcomes, B and C. Notice that B and C are classical in this setting.

By the nature of quantum measurements [69], the outcomes B and C may change from one run of the experiment to another, even when the same measurements X and Y are taken. The outcomes are governed by a conditional probability mass function $P_{BC|XY}(b, c|x, y)$, and can be estimated by running the experiment for a sufficient number of rounds. The function $P_{BC|XY}$ is also called a behavior, or, *a correlation*. In general, the correlation cannot necessarily be separated as $P_{B|X} \times P_{C|Y}$, even when the observers are remote. This does not necessarily imply a direct influence of one system on the other.

The notion of locality refers to a situation where past factors can be encapsulated in some random variable U, also referred to as a hidden variable [102, 103], such that when taking it into account, the correlation between the outcomes is broken, i.e.,

$$P_{BC|XY}(b, c|x, y) = \int_{\text{supp}(U)} p_U(u) P_{B|XU}(b|x, u) P_{C|YU}(c|y, u)\, \mathrm{d}u\,. \tag{68}$$

The predictions of the quantum theory for certain settings involving quantum entanglement do not follow the locality condition in (68). Suppose that Bob and Charlie share a bipartite state $\rho_{M_1M_2}$. As they perform local measurements $\{F_{b|x}\,,\ b \in \mathcal{B}\}$ and $\{D_{c|y}\,,\ c \in \mathcal{C}\}$, they generate the following correlation:

$$P_{BC|XY}(b, c|x, y) = \mathrm{Tr}\left[\left(F_{b|x} \otimes D_{c|y}\right) \rho_{M_1M_2}\right]\,. \tag{69}$$

The set of all quantum correlations is often denoted in the literature by C_q [104].

One of the simplest experiments demonstrating nonlocal behavior is the CHSH setting, named after Clauser, Horne, Shimony, and Holt [105]. Consider the Bell experiment setting shown in Figure 17 (a), where the observers Bob and Charlie can only perform one of two measurements, $X, Y \in \{0, 1\}$. The outcomes are limited to two values as well $B, C \in \{\pm 1\}$. Consider

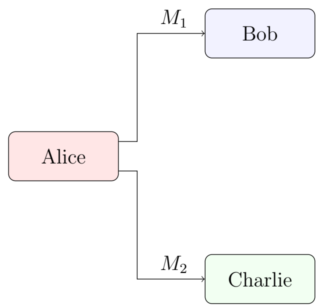
$$S = \langle B_0C_0\rangle + \langle B_0C_1\rangle + \langle B_1C_0\rangle - \langle B_1C_1\rangle\,. \tag{70}$$

where $\langle B_xC_y\rangle$ are the corresponding expectation values, $\langle B_xC_y\rangle = \sum_{b,c\in\{\pm1\}} bc \cdot P_{BC|XY}(b, c|x, y)$, for $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

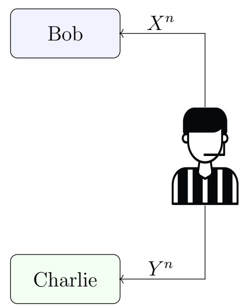
If the correlation $P_{BC|XY}$ satisfies the locality condition in (68), then $S \leq 2$ must hold [106]. However, in the quantum case, this inequality may be violated. Suppose Bob and Charlie are each provided with a qubit from an EPR pair $|\Phi_{M_1M_2}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Denote the Pauli operators by $(\Sigma_1, \Sigma_2, \Sigma_3)$. Bob and Charlie choose their measurements depending on the values of X and Y, respectively. If $X = 0$, Bob measures the Σ_3 observable. Otherwise, if $X = 1$, he measures the Σ_1 observable. As for Charlie, if $Y = 0$, he measures the observable $\frac{-\Sigma_3 - \Sigma_1}{\sqrt{2}}$, and if $Y = 1$, he measures $\frac{\Sigma_3 - \Sigma_1}{\sqrt{2}}$. This yields $S = 2\sqrt{2} > 2$ (see (70)), demonstrating the nonlocal nature of quantum entanglement. Based on this violation, quantum correlations cannot be explained using the theory of classical hidden variables [106].

B. Refereed games

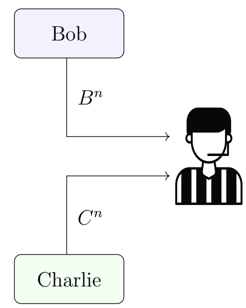
Refereed games can be viewed as another representation of the Bell setting. Specifically, consider the refereed game in Figure 17 (b). The referee provides two questions $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, according to some probability distribution $p_{X,Y}$. He



(a) Phase 1.



(b) Phase 2.



(c) Phase 3.

Fig. 18. Implementation of a refereed game in three phases: (a) In Phase 1, the source (Alice) distributes the correlation resources between the players (Bob and Charlie). (b) In Phase 2, the referee generates the questions, and sends the respective question to Bob and Charlie. (c) In Phase 3, upon receiving their questions, Bob and Charlie produce their answers and inform the referee. Once the referee is informed, he decides whether the game is won.

sends X to the first player (Bob), and Y to the second (Charlie). Upon receiving their question, Bob and Charlie respond with classical answers $B \in \mathcal{B}$ and $C \in \mathcal{C}$, respectively. We note that the alphabets $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{B}$, and $\mathcal{C}$ are assumed to be finite. The referee decides that the game is won if the realization of the tuple (X, Y, B, C) satisfies a specific condition $\mathscr{W}$, set by the rules of the game. This condition is represented by an indicator function,

$$V(x,y,b,c) = \begin{cases} 1 & \text{If } (x,y,b,c) \text{ satisfy } \mathscr{W}\,, \\ 0 & \text{otherwise}\,\,. \end{cases} \tag{71}$$

We refer to the procedure above as a single-shot game. We now discuss the game implementation and rules.

1) Resources: As in the Bell setting, a source distributes correlated physical systems before the procedure begins (see Figure 17 (a)). Here, we refer to the source of the correlation resources as *Alice*.

2) Strategy: Before the game starts, i.e., before the referee has chosen his questions, Alice, Bob, and Charlie meet and agree on a game strategy and the required correlation resources. The optimal game strategy and the required correlations for the strategy implementation depend on the game rules.

3) No signaling: During the course of the game, Bob and Charlie cannot communicate with each other. They can, however, exploit the correlation resources in order to coordinate their answers through quantum measurements.

We can also give an equivalent description of the game implementation in terms of three phases. In Phase 1, the source (Alice) distributes the correlation resources, M_1 and M_2, between the players (Bob and Charlie). In Phase 2, the referee generates the question pair (x, y) according to p_{XY}, and sends x and y them to Bob and Charlie, respectively. In Phase 3, upon receiving their questions, Bob and Charlie produce their answers, B and C. Once the referee is informed, he decides whether the game is won. We refer to this description as a single shot game.

The winning probability is thus

$$\pi(P_{BC|XY}) = \sum_{(x,y,b,c)\in\mathcal{X}\times\mathcal{Y}\times\mathcal{B}\times\mathcal{C}} p_{XY}(x,y) P_{BC|XY}(b,c|x,y) \cdot V(x,y,b,c)\,. \tag{72}$$

The performance depends directly on the correlation $P_{BC|XY}$ that Alice, Bob, and Charlie simulate as a consequence of the three phases above. For example, in the CHSH game, the winning condition is $x \wedge y = b \oplus c$., where $x, y, b, c \in \{0, 1\}$. $\pi(P_{BC|XY}) = \frac{1}{2}\left(1 + \frac{S}{4}\right)$. Classical strategies may generate correlations $P_{BC|XY}$ such that $S \leq 2$ (see (70)), hence, the game can be won with probability $\pi(P_{BC|XY}) \leq 0.75$. Whereas, entanglement allows for $S = 2\sqrt{2}$, for which $\pi(P_{BC|XY}) = 0.8535$.

C. Sequential game

We now introduce a sequential version of the refereed game, see Figure 18. In Phase 1, the source (Alice) distributes the correlation resources, M_1 and M_2, between the players (Bob and Charlie). In Phase 2, the referee generates a sequence of n independent question pairs (x_i, y_i) according to p_{XY}, and sends x^n and y^n to Bob and Charlie, respectively. In Phase 3, the players produce their responses. Bob and Charlie choose their measurements depending on x^n and y^n, respectively. Then, they perform their respective measurements on M_1 and M_2. They send the measurement outcomes b^n and c^n, respectively, to the referee. The worst-case winning probability is thus

$$\pi_n(\widehat{P}_{B^nC^n|X^nY^n}) = \min_{i\in[n]} \sum_{(x^n,y^n,b^n,c^n)\in\mathcal{X}^n\times\mathcal{Y}^n\times\mathcal{B}^n\times\mathcal{C}^n} p^n_{XY}(x^n,y^n)\widehat{P}_{B^nC^n|X^nY^n}(b^n,c^n|x^n,y^n)\cdot V(x_i,y_i,b_i,c_i)\,, \tag{73}$$

Notice that if we simulate a product correlation, i.e., $\widehat{P}_{B^nC^n|X^nY^n} \approx P^n_{BC|XY}$, then

$$\pi_n(\widehat{P}_{B^nC^n|X^nY^n}) \approx \pi(P_{BC|XY})\,. \tag{74}$$

D. Coordination as part of a game strategy

We now present the connection between quantum coordination and refereed games explicitly. We now insert a broadcast coordination scheme into the game strategy. We consider the special case where B and C are classical, while A is null (say, $\dim(\mathcal{H}_A) = 1$).

Consider the sequential game setup described in Figure 18. In Phase 1, the source (Alice) prepares the quantum resources M_1 and M_2 using the coordination encoding map $\mathcal{E}$. She then distributes the resources between the respective players (Bob and Charlie), using noiseless quantum links at rates Q_1 and Q_2. In Phase 2, the referee chooses question sequences X^n and Y^n that have no correlation with the quantum resources, as in the broadcast network model. In Phase 3, Bob and Charlie use the encoding measurements $\mathcal{F}_{X^nM_1\to B^n}$ and $\mathcal{D}_{Y^nM_2\to C^n}$. They obtain B^n and C^n as measurement outcomes and inform the referee.

This coordination strtegy generates a classical-correlation state, $\widehat{\rho}_{X^nY^nB^nC^n} \approx \omega^{\otimes n}_{XYBC}$, where

$$\omega_{XYBC} = \sum_{(x,y,b,c)\in\mathcal{X}\times\mathcal{Y}\times\mathcal{B}\times\mathcal{C}} p_{XY}(x,y)P_{BC|XY}(b,c|x,y)\,|x,y,b,c\rangle\!\langle x,y,b,c|\ , \tag{75}$$

which leads to a winning probability $\pi(P_{BC|XY})$ (see (72)).

Let $\mathscr{S}(\gamma)$ denote the set of correlations $P_{BC|XY}$ that win the game with probability of at least γ. Based on our results, the game can be won with probability γ if and only if Alice can send qubits to Bob and Charlie at rates Q_1 and Q_2 that satisfy the constraints in Theorem 8 with respect to some correlation $P_{BC|XY} \in \mathscr{S}(\gamma)$. Being able to generate entanglement between Bob and Charlie, can provide an advantage by inducing quantum correlations stronger than their classical counterparts, hence allowing for higher winning probabilities (see Subsection VII-A).

E. Example: The CHSH game

A well known game demonstrating the advantage of nonlocal correlations is the CHSH game. Suppose that the players first simulate the following state using a broadcast coordination code:

$$\left|\omega^{(x,y)}\right\rangle = \sqrt{\alpha_{x,y}}\,|00\rangle + \sqrt{1-\alpha_{x,y}}\,|11\rangle\ , \tag{76}$$

where $\alpha_{x,y}$ are given parameters in $[0,1]$, for $(x,y) \in \mathcal{X}\times\mathcal{Y}$. Applying the same measurement strategy as in the CHSH experiment in Subsection VII-A, we obtain a correlation P such that the winning probability is given by

$$\pi^{\text{CHSH}}(P) = \frac{1}{16}\sum_{x,y\in\{0,1\}}\left[\frac{1+2\sqrt{2}}{\sqrt{2}} + \sqrt{2\alpha_{x,y}(1-\alpha_{x,y})}\right]. \tag{77}$$

For $\alpha \equiv \frac{1}{2}$, i.e., with maximal Bell violation [107], we get a maximal winning probability of $\pi^{\text{CHSH}}(P) = 0.8535$. For $\alpha \equiv 0$, when there is no correlation, we have $\pi^{\text{CHSH}}(P) = 0.6767$. In this case, the CHSH measurement strategy is even worse that the best classical strategy, for which $\pi^{\text{CHSH}}(P) = 0.75$.

By Theorem 8, Phase 1 requires the communication rates $Q_1 \geq \frac{1}{2}H_2\left(\frac{1}{2}\left(\alpha_{0,0}+\alpha_{0,1}\right)\right) + \frac{1}{2}H_2\left(\frac{1}{2}\left(\alpha_{1,0}+\alpha_{1,1}\right)\right)$ and $Q_2 \geq \frac{1}{2}H_2\left(\frac{1}{2}\left(\alpha_{0,0}+\alpha_{1,0}\right)\right) + \frac{1}{2}H_2\left(\frac{1}{2}\left(\alpha_{0,1}+\alpha_{1,1}\right)\right)$, where $H_2(\cdot)$ is the binary entropy function. In particular, for a constant parameter, $\alpha_{x,y} = \alpha$ for all x, y, we have $Q_j \geq H_2(\alpha)$. There is a threshold value α^* for which the CHSH measurement strategy has the same performance as the best classical strategy. Specifically, we obtain a Bell violation provided that $\alpha_{x,y} > 0.04491$ for all $(x,y) \in \mathcal{X}\times\mathcal{Y}$. The winning probability for a constant parameter α, is shown in Figure 19. Since the gradient is unbounded near $\alpha = 0$, even a small amount of entanglement can have a significant effect on the winning probability. As we approach $\alpha = \frac{1}{2}$, the gradient diminishes. The Bell violation threshold requires $Q_j \geq H_2(\alpha^*) = 0.2643$.

To summarize, we have discussed the notion of Bell experiments and their direct connection to the simulation of nonlocal correlations. We then discussed refereed games in the standard single-shot form and the sequential form. We have shown that coordination in the broadcast network can be viewed as the overall game strategy, i.e., the preparation of the pre-shared resources (Phase 1) and the measurement (Phase 3). In this sense, coordination is the enabler of quantum strategies that achieve higher winning probabilities compared to classical ones.

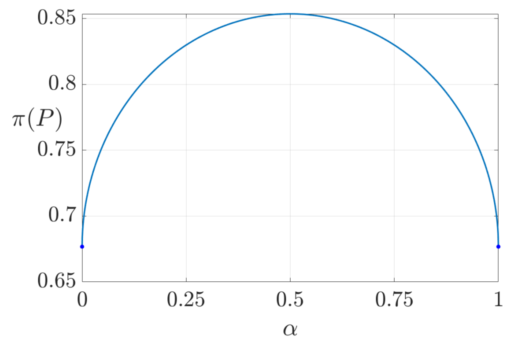


Fig. 19. Winning probability as a function of α.

Quantum coordination can be useful for generating nontrivial correlations in other types of games as well. For example, in pseudo-telepathy games, quantum strategies guarantee winning with probability 1. One example is the magic square game [108], where (X, Y) are the coordinates of a cell in the square, and the players win the game if they can provide 3 bits each that satisfy a parity condition. In this case, the game can be won with $Q_j = 2$ qubits per question, for each player. Slofstra and Vidick [109] presented a game where coordination of a correlation that could win with probability $(1 - e^{-T})$ requires $Q_j \propto T$ qubits per question.

VIII. Two Node Analysis (Classical Links)

Consider the two node network in Figure 10. Our proof for Theorem 1 is based on quantum resolvability [92].

Theorem 10 (see [55, 92, 93]). Consider an ensemble, $\{p_X, \rho_A^x\}_{x\in\mathcal{X}}$, and a random codebook that consists of 2^{nR} independent sequence, $X^n(m)$, $m \in [2^{nR}]$, each is i.i.d. $\sim p_X$. If $R > I(X;A)_\rho$, then for every $\delta > 0$ and a sufficiently large n,

$$\mathbb{E}\left[\left\|\rho_A^{\otimes n} - \frac{1}{2^{nR}}\sum_{m=1}^{2^{nR}} \rho_{A^n}^{X^n(m)}\right\|_1\right] \le \delta\,, \tag{78}$$

where $\rho_{A^n}^{x^n} \equiv \bigotimes_{k=1}^n \rho_A^{x_k}$, and the expectation is over all realizations of the random codebook.

A. Achievability proof

Assume (R_0, R_1) is in the interior of $\mathcal{R}_{\text{2-node}}(\omega)$. We need to construct a code that consists of an encoding channel $F(m_1|x^n, m_0)$ and a c-q decoding channel $\mathcal{D}_{M_0M_1\to B^n}$, such that the error requirement in (19) holds.

By the definition of $\mathscr{S}_{\text{2-node}}(\omega)$, there exists an ensemble $\{p_U(u)p_{X|U}(x|u)\,,\ \theta_B^u\}$, with an average c-c-q state,

$$\sigma_{UXB} = \sum_{u\in\mathcal{U}} p_U\,(u)\,|u\rangle\langle u|_U \otimes \sigma_{XB}^u \tag{79}$$

such that

$$\sigma_{XB}^u = \sum_{x\in\mathcal{X}} p_{X|U}\,(x|u)\,|x\rangle\langle x|_X \otimes \theta_B^u\,,\ u \in \mathcal{U} \tag{80}$$

$$\sigma_{XB} = \omega_{XB}\,, \tag{81}$$

$$R_1 > I(X;U)_\sigma\,,\ R_0 + R_1 > I(XB;U)_\sigma\,. \tag{82}$$

Classical codebook generation: Select a random codebook $\mathscr{C} = \{u^n(m_0, m_1)\}$ by drawing $2^{n(R_0+R_1)}$ i.i.d. sequences according to the distribution $p_U^n\,(u^n) = \prod_{k=1}^n p_U\,(u_k)$. Reveal the codebook to Alice and Bob.

Let (m_0, m_1) be a pair of random indices, uniformly distributed over $[2^{nR_0}] \times [2^{nR_1}]$. Define the following PMF

$$\widetilde{P}_{X^nM_0M_1}(x^n, m_0, m_1) \equiv \frac{1}{2^{n(R_0+R_1)}} p_{X|U}^n\,(x^n|u^n\,(m_0, m_1))\,. \tag{83}$$

Encoder: We define the encoding channel F as the conditional distribution above, i.e., $F = \widetilde{P}_{M_1|X^nM_0}$.

Decoder: As Bob receives m_1 from Alice, and the random element m_0, he prepares the output state $\mathcal{D}_{M_0M_1\to B^n}(m_0, m_1) = \theta_{B^n}^{u^n(m_0,m_1)}$.

***Error analysis**:* Let $\delta > 0$. Consider a fixed realization m_0 of the random element. Given $M_0 = m_0$, the encoder sends $m_1 \sim F(\cdot|x^n, m_0)$. By the classical resolvability theorem Cuff [45] has shown that $R_1 > I(X;U)_\sigma$ guarantees

$$\mathbb{E}\left\|\widetilde{P}_{M_0X^n} - p_{M_0} \times p_X^n\right\|_1 \leq \delta \tag{84}$$

for a sufficiently large n, where $\widetilde{P}_{M_0X^n}$ is as in (83). Recall that $\widetilde{P}_{M_0X^n}$ is random, since the codebook $\mathscr{C}$ is random. Hence, the expectation is over all realizations of $\mathscr{C}$. The resulting state is

$$\widehat{\rho}_{X^nB^n} = \frac{1}{2^{nR_0}} \sum_{m_0,x^n} \left(p_X^n(x^n)\,|x^n\rangle\!\langle x^n|_{X^n} \otimes \sum_{m_1\in[2^{nR_1}]} \widetilde{P}_{M_1|X^nM_0}(m_1|x^n,m_0)\theta_{B^n}^{u^n(m_0,m_1)}\right) \tag{85}$$

According to (84), the probability distributions $\widetilde{P}_{M_0,X^n}$ and $p_{M_0} \times p_X^n$ are close on average. Then, let

$$\widehat{\tau}_{X^nB^n} \equiv \sum_{m_0,x^n} \widetilde{P}_{M_0X^n}(m_0x^n)\,|x^n\rangle\!\langle x^n|_{X^n} \otimes \sum_{m_1\in[2^{nR_1}]} \widetilde{P}_{M_1|X^nM_0}(m_1|x^n,m_0)\theta_{B^n}^{u^n(m_0,m_1)}\,. \tag{86}$$

By (84), it follows that

$$\mathbb{E}\left\|\widehat{\tau}_{X^nB^n} - \widehat{\rho}_{X^nB^n}\right\|_1 \leq \delta\,. \tag{87}$$

Observe that

$$\begin{aligned}
\widehat{\tau}_{X^nB^n} &= \sum_{m_0,m_1,x^n} \widetilde{P}_{M_0M_1X^n}(m_0,m_1,x^n)\,|x^n\rangle\!\langle x^n|_{X^n} \otimes \theta_{B^n}^{u^n(m_0,m_1)} \\
&= \frac{1}{2^{n(R_0+R_1)}} \sum_{m_0,m_1,x^n} p^n_{X|U}(x^n|u^n(m_0,m_1))\,|x^n\rangle\!\langle x^n|_{X^n} \otimes \theta_{B^n}^{u^n(m_0,m_1)} \\
&= \frac{1}{2^{n(R_0+R_1)}} \sum_{m_0,m_1} \sigma_{X^nB^n}^{u^n(m_0,m_1)}
\end{aligned} \tag{88}$$

where the second equality is due to the definition of $\widetilde{P}$ in (83), and the last line follows from (80).

Thus, according to the quantum resolvability theorem, Theorem 10, when applied to the joint system XB, for $R_0 + R_1 > I(XB;U)_\sigma$, we have

$$\mathbb{E}\left\|\sigma_{XB}^{\otimes n} - \widehat{\tau}_{X^nB^n}\right\|_1 \leq \delta \tag{89}$$

for a sufficiently large n. Therefore, by the triangle inequality,

$$\begin{aligned}
\mathbb{E}\left\|\omega_{XB}^{\otimes n} - \widehat{\rho}_{X^nB^n}\right\|_1 &\leq \mathbb{E}\left\|\omega_{XB}^{\otimes n} - \widehat{\tau}_{X^nB^n}\right\|_1 + \mathbb{E}\left\|\widehat{\tau}_{X^nB^n} - \widehat{\rho}_{X^nB^n}\right\|_1 \\
&\leq 2\delta
\end{aligned} \tag{90}$$

by (81), (87) and (89). $\square$

B. Converse proof

Let (R_0, R_1) be an achievable rate pair. Then, there exists a sequence $\left(2^{nR_0}, 2^{nR_1}, n\right)$ of coordination codes such that the joint quantum state $\widehat{\rho}_{X^nB^n}$ satisfies

$$\left\|\omega_{XB}^{\otimes n} - \widehat{\rho}_{X^nB^n}\right\|_1 \leq \varepsilon_n \tag{91}$$

where ε_n tends to zero as $n \to \infty$.

Fix an index $i \in \{1, \ldots, n\}$. By trace monotonicity [69], taking the partial trace over X_j, B_j, $j \neq i$, maintains the inequality. Thus,

$$\left\|\omega_{XB} - \widehat{\rho}_{X_iB_i}\right\|_1 \leq \varepsilon_n\,. \tag{92}$$

Then, by the AFW inequality [110],

$$\left|H\left(X^nB^n\right)_{\widehat{\rho}} - nH\left(XB\right)_\omega\right| \leq n\beta_n\,, \tag{93}$$

and

$$\left|H\left(X_iB_i\right)_{\widehat{\rho}} - H\left(XB\right)_\omega\right| \leq \beta_n\,, \tag{94}$$

for $i \in [n]$, where β_n tends to zero as $n \to \infty$. Therefore,

$$\begin{aligned}\left| H\left(X^n B^n\right)_{\widehat{\rho}} - \sum_{i=1}^{n} H\left(X_i B_i\right)_{\widehat{\rho}} \right| &\leq \left| H\left(X^n B^n\right)_{\widehat{\rho}} - nH\left(XB\right)_{\omega} \right| + \left| nH\left(XB\right)_{\omega} - \sum_{i=1}^{n} H\left(X_i B_i\right)_{\widehat{\rho}} \right| \\ &\leq \left| H\left(X^n B^n\right)_{\widehat{\rho}} - nH\left(XB\right)_{\omega} \right| + \sum_{i=1}^{n} \left| H\left(XB\right)_{\omega} - H\left(X_i B_i\right)_{\widehat{\rho}} \right| \\ &\leq 2n\beta_n \,. \end{aligned} \tag{95}$$

Now, we have

$$n(R_0 + R_1) \geq H(M_0 M_1) \tag{96}$$
$$\geq I(X^n B^n; M_0 M_1)_{\widehat{\rho}} \tag{97}$$

since the conditional entropy is nonnegative for classical and c-q states, and the CR element M_0 is statistically independent of the source X^n. Furthermore, by entropy sub-additivity [69],

$$\begin{aligned}I(X^n B^n; M_0 M_1)_{\widehat{\rho}} &\geq H(X^n B^n)_{\widehat{\rho}} - \sum_{i=1}^{n} H(X_i B_i | M_0 M_1)_{\widehat{\rho}} \\ &\geq \sum_{i=1}^{n} I(X_i B_i; M_0 M_1)_{\widehat{\rho}} - 2n\beta_n \end{aligned} \tag{98}$$

where the last inequality follows from (95). Defining a time-sharing variable $I \sim \text{Unif}[n]$, this can be written as

$$R_0 + R_1 + 2\beta_n \geq I(X_I B_I; M_0 M_1 | I)_{\widehat{\rho}} \tag{99}$$

with respect to the extended state:

$$\widehat{\rho}_{I M_0 M_1 X_I B_I} = \frac{1}{n} \sum_{i=1}^{n} |i\rangle\langle i| \otimes \widehat{\rho}_{M_0 M_1 X_i B_i} \,. \tag{100}$$

Observe that by (92) and the triangle inequality,

$$\begin{aligned}\left\| \omega_{XB} - \widehat{\rho}_{X_I B_I} \right\|_1 &= \left\| \omega_{XB} - \frac{1}{n} \sum_{i=1}^{n} \widehat{\rho}_{X_i B_i} \right\|_1 \\ &\leq \varepsilon_n \,. \end{aligned} \tag{101}$$

Thus, by the AFW inequality,

$$\begin{aligned}I(X_I B_I; I)_{\widehat{\rho}} &= H(X_I B_I)_{\widehat{\rho}} - \frac{1}{n} \sum_{i=1}^{n} H(X_i B_i)_{\widehat{\rho}} \\ &\leq \gamma_n \,, \end{aligned} \tag{102}$$

where γ_n tends to zero. Together with (99), it follows that

$$R_0 + R_1 + 2\beta_n + \gamma_n \geq I(X_I B_I; M_0 M_1 I)_{\widehat{\rho}} \tag{103}$$

By similar arguments,

$$R_1 + 2\beta_n + \gamma_n \geq I(X_I; M_0 M_1 I) \tag{104}$$

To complete the converse proof, we identify U, X, and B with (M_0, M_1, I), X_I, and B_I, respectively. Observe that given (m_0, m_1, i), the joint state of X_I and B_I is $\left(\sum_{x_i \in \mathcal{X}} p_{X_i | M_0 M_1}(x_i | m_0, m_1) \, |x_i\rangle\langle x_i|_{X_I}\right) \otimes \rho_{B_i}^{(m_0, m_1)}$, where $p_{X^n | M_0 M_1}$ is the a posteriori probability distribution. Thus, there X and B are uncorrelated when conditioned on U, as required.

The bound on $|\mathcal{U}|$ follows by applying the Caratheodory theorem to the real-valued parametric representation of density matrices, as in [111, App. B]. $\square$

IX. No-Communication Analysis

Consider the no-communication network in Figure 11, of a quantum state ω_{ABC}. To prove Theorem 4, we use similar tools. The achievability proof is straightforward, and it is thus omitted.

Then, consider the converse part. Assume that R_0 is achievable. Therefore, there exists a sequence of $(2^{nR_0}, n)$ of coordination codes such that for a sufficiently large values of n,

$$\left\|\widehat{\rho}_{A^nB^nC^n} - \omega_{ABC}^{\otimes n}\right\|_1 \leq \varepsilon_n\,, \tag{105}$$

where $\varepsilon_n \to 0$ as $n \to \infty$.

Applying the chain rule,

$$\begin{aligned} nR_0 &\geq H(M_0) &(106)\\ &\geq I(A^nB^nC^n; M_0)_{\widehat{\rho}} &(107)\\ &= \sum_{i=1}^{n} I(A_iB_iC_i; M_0|A^{i-1}B^{i-1}C^{i-1})_{\widehat{\rho}} &(108)\end{aligned}$$

For every $i \in [n]$, by trace monotonicity [69],

$$\left\|\omega_{ABC}^{\otimes i} - \widehat{\rho}_{A^iB^iC^i}\right\|_1 \leq \varepsilon_n\,. \tag{109}$$

Then, by the AFW inequality [110] [69, Ex. 11.10.2],

$$\left|I(A_iB_iC_i; A^{i-1}B^{i-1}C^{i-1})_{\widehat{\rho}} - I(A_iB_iC_i; A^{i-1}B^{i-1}C^{i-1})_{\omega^{\otimes i}}\right| \leq \beta_n\,, \tag{110}$$

where β_n tends to zero as $n \to \infty$. That is,

$$I(A_iB_iC_i; A^{i-1}B^{i-1}C^{i-1})_{\widehat{\rho}} \leq \beta_n \tag{111}$$

since $A_iB_iC_i$ and $(A_jB_jC_j)_{j<i}$ are in a product state $\omega \otimes \omega^{\otimes(k-1)}$. Hence, by (108),

$$\begin{aligned} nR_0 &\geq \sum_{i=1}^{n} I(A_iB_iC_i; M_0A^{i-1}B^{i-1}C^{i-1})_{\widehat{\rho}} - n\beta_n\\ &\geq \sum_{i=1}^{n} I(A_iB_iC_i; M_0)_{\widehat{\rho}} - n\beta_n\\ &\geq n\left(\inf_{\sigma_{UABC} \in \mathscr{S}_{NC}(\omega)} I(U; ABC)_\sigma - 2\beta_n\right) &(112)\end{aligned}$$

taking $U \equiv M_0$, as the encoders are uncorrelated given M_0. $\square$

X. Broadcast Analysis (Classical Links)

Consider coordination in broadcast network, as in Figure 12 in the main text, of a c-q-q state ω_{XBC}. To prove the capacity theorem, Theorem 5, we use similar tools as in Section VIII.

A. Achievability proof

Assume (R_0, R_1) is in the interior of $\mathcal{R}_{\text{BC}}(\omega)$. We need to construct a code that consists of an encoding channel $F(m_1|x^n, m_0)$ and a two c-q decoding channels $\mathcal{D}_{M_0M_1\to B^n}$ and $\mathcal{D}_{M_0M_1\to C^n}$,such that

$$\begin{aligned}\Bigg\|\omega_{XB}^{\otimes n} - \frac{1}{2^{nR_0}} \sum_{m_0\in[2^{nR_0}]} \sum_{x^n\in\mathcal{X}^n} p_X^n(x^n)\, |x^n\rangle\!\langle x^n|_{X^n} \otimes\\ \sum_{m_1\in[2^{nR_1}]} F(m_1|x^n, m_0)\mathcal{D}_{M_0M_1\to B^n}(m_1, m_0) \otimes \mathcal{D}_{M_0M_1\to C^n}(m_1, m_0)\Bigg\|_1 \leq \varepsilon\,. \end{aligned} \tag{113}$$

According to the definition of $\mathscr{S}_{\text{BC}}(\omega)$ (see Subsection V-C), there exists a c-c-q-q state σ_{XUBC} that can be written as

$$\sigma_{XUBC} = \sum_{(x,u)\in\mathcal{X}\times\mathcal{U}} p_{XU}(x,u)\, |x\rangle\!\langle x|_X \otimes |u\rangle\!\langle u|_U \otimes \theta_B^u \otimes \eta_C^u \tag{114}$$

and satisfy

$$\sigma_{XBC} = \omega_{XBC} \tag{115}$$

We will also consider conditioning on $U = u$, and denote

$$\sigma_{XBC}^u = \sum_{x\in\mathcal{X}} p_{X|U}(x|u)\, |x\rangle\!\langle x|_X \otimes \theta_B^u \otimes \eta_C^u\,. \tag{116}$$

Classical codebook generation: Select a random codebook $\mathscr{C}_{\text{BC}} = \{u^n(m_0, m_1)\}$ by drawing $2^{n(R_0+R_1)}$ i.i.d. sequences according to the distribution p_U^n. Reveal the codebook.

Encoder: Define the encoding channel as $F = \widetilde{P}_{M_1|X^n M_0}$, where $\widetilde{P}_{X^n M_0 M_1}$ be a joint distribution as in (83).

Decoders: As Bob and Charlie receive m_1 from Alice, and the random element m_0, they prepare the following output states,

$$\mathcal{D}^{(1)}_{M_0 M_1 \to B^n}(m_0, m_1) = \theta_B^{u^n(m_0,m_1)}, \tag{117}$$

$$\mathcal{D}^{(2)}_{M_0 M_1 \to C^n}(m_0, m_1) = \eta_C^{u^n(m_0,m_1)}. \tag{118}$$

Error analysis: Let $\delta > 0$. The encoder sends $m_1 \sim F(\cdot|x^n, m_0)$. As in Subsection VIII-A, given m_0, if $R_1 > I(X;U)$, then

$$\mathbb{E}\left\|\widetilde{P}_{M_0 X^n} - p_{M_0} \times p_X^n\right\|_1 \leq \delta \tag{119}$$

for a sufficiently large n. As $\widetilde{P}_{M_0 X^n}$ depends on the random codebook $\mathscr{C}_{\text{BC}}$, the expectation is over all realizations of $\mathscr{C}_{\text{BC}}$. The resulting state is

$$\begin{aligned}
&\widehat{\rho}_{X^n B^n C^n} \\
&= \frac{1}{2^{nR_0}} \sum_{m_0 \in [2^{nR_0}]} \sum_{x^n \in \mathcal{X}^n} \Big(p_X^n(x^n) \, |x^n\rangle\!\langle x^n|_{X^n} \otimes \sum_{m_1 \in [2^{nR_1}]} F(m_1|x^n, m_0) \mathcal{D}_{M_0 M_1 \to B^n}(m_0, m_1) \otimes \mathcal{D}_{M_0 M_1 \to C^n}(m_0, m_1) \Big) \\
&= \frac{1}{2^{nR_0}} \sum_{m_0, x^n} \Big(p_X^n(x^n) \, |x^n\rangle\!\langle x^n|_{X^n} \otimes \sum_{m_1 \in [2^{nR_1}]} \widetilde{P}_{M_1|X^n M_0}(m_1|x^n, m_0) \theta_B^{u^n(m_0,m_1)} \otimes \eta_C^{u^n(m_0,m_1)} \Big).
\end{aligned} \tag{120}$$

According to (119), the probability distributions $\widetilde{P}_{M_0 X^n}$ and $p_{M_0} \times p_X^n$ are close on average. Then, let

$$\widehat{\tau}_{X^n B^n C^n} \equiv \sum_{m_0, x^n} \widetilde{P}_{M_0 X^n}(m_0, x^n) \, |x^n\rangle\!\langle x^n|_{X^n} \otimes \sum_{m_1 \in [2^{nR_1}]} \widetilde{P}_{M_1|X^n M_0}(m_1|x^n, m_0) \theta_B^{u^n(m_0,m_1)} \otimes \eta_C^{u^n(m_0,m_1)}. \tag{121}$$

Then, it follows that

$$\mathbb{E}\left\|\widehat{\tau}_{X^n B^n C^n} - \widehat{\rho}_{X^n B^n C^n}\right\|_1 \leq \delta, \tag{122}$$

by (119). Observe that

$$\begin{aligned}
\widehat{\tau}_{X^n B^n C^n} &= \sum_{m_0, m_1, x^n} \widetilde{P}_{M_0 M_1 X^n}(m_0, m_1, x^n) \, |x^n\rangle\!\langle x^n|_{X^n} \otimes \theta_B^{u^n(m_0,m_1)} \otimes \eta_C^{u^n(m_0,m_1)} \\
&= \frac{1}{2^{n(R_0+R_1)}} \sum_{m_0, m_1, x^n} p^n_{X|U}(x^n|u^n(m_0, m_1)) \, |x^n\rangle\!\langle x^n|_{X^n} \otimes \theta_B^{u^n(m_0,m_1)} \otimes \eta_C^{u^n(m_0,m_1)} \\
&= \frac{1}{2^{n(R_0+R_1)}} \sum_{m_0, m_1} \sigma_{X^n B^n C^n}^{u^n(m_0,m_1)},
\end{aligned} \tag{123}$$

where the second equality is due to the definition of $\widetilde{P}$ in (83), and the last line follows from (116).

Thus, according to the quantum resolvability theorem 10, when applied to the joint system XBC, we have

$$\mathbb{E}\left\|\sigma^{\otimes n}_{X B^n C^n} - \widehat{\tau}_{X^n B^n C^n}\right\|_1 \leq \delta \tag{124}$$

for a sufficiently large n. Therefore, by the triangle inequality,

$$\begin{aligned}
\mathbb{E}\left\|\omega^{\otimes n}_{XBC} - \widehat{\rho}_{X^n B^n C^n}\right\|_1 &\leq \mathbb{E}\left\|\omega^{\otimes n}_{X B^n C^n} - \widehat{\tau}_{X^n B^n C^n}\right\|_1 + \mathbb{E}\left\|\widehat{\tau}_{X^n B^n C^n} - \widehat{\rho}_{X^n B^n C^n}\right\|_1 \\
&\leq 2\delta
\end{aligned} \tag{125}$$

by (115), (122) and (124). $\square$

B. Converse proof

Let (R_0, R_1) be an achievable coordination rate pair for the simulation of a c-q-q state ω_{XBC} in the broadcast setting. Then, there exists a sequence of $\left(2^{nR_0}, 2^{nR_1}, n\right)$ coordination codes such that the joint quantum state $\widehat{\rho}_{X^n B^n C^n}$ satisfies

$$\left\|\omega^{\otimes n}_{X B^n C^n} - \widehat{\rho}_{X^n B^n C^n}\right\|_1 \leq \varepsilon_n, \tag{126}$$

where ε_n tends to zero as $n \to \infty$. Based on the same arguments as for the two-node network (see Subsection VIII-B), we have

$$R_0 + R_1 + 2\beta_n + \gamma_n \geq I(X_I B_I C_I; M_0 M_1 I)_{\widehat{\rho}}\,, \tag{127}$$
$$R_1 + 2\beta_n + \gamma_n \geq I(X_I; M_0 M_1) \tag{128}$$

with respect to the extended state

$$\widehat{\rho}_{I M_0 M_1 X_I B_I C_I} = \frac{1}{n} \sum_{i=1}^{n} |i\rangle\langle i| \otimes \widehat{\rho}_{M_0 M_1 X_i B_i C_i}\,. \tag{129}$$

To complete the converse proof, we identify U, X, and BC with (M_0, M_1, I), X_I, and $B_I C_I$, respectively. Observe that given (m_0, m_1, i), the joint state of X_I, B_I and C_I is

$$\left(\sum_{x_i \in \mathcal{X}} p_{X_i|M_0 M_1}(x_i|m_0, m_1)\, |x_i\rangle\langle x_i|_{X_I} \right) \otimes \rho_{B_I}^{(m_0,m_1)} \otimes \rho_{C_I}^{(m_0,m_1)}\,, \tag{130}$$

where $p_{X^n|M_0 M_1}$ is the a posteriori probability distribution. Thus, there is no correlation between X, B, and C when conditioned on U, as required. $\square$

XI. Cascade Network Analysis (Quantum Links)

We prove the rate characterization in Theorem 6. Consider the cascade network in Figure 13.

A. Achievability proof

The proof for the direct part exploits the state redistribution result by Yard and Devetak in [17]. We first describe the state redistribution problem. Consider two parties, Alice and Bob. Let their systems be described by the joint state ψ_{ABG}, where A and B belong to Alice, and G belongs to Bob. Let the state $|\psi_{ABGR}\rangle$ be a purification of ψ_{ABG}. Alice and Bob would like to redistribute the state ψ_{ABG} such that B is transferred from Alice to Bob. Alice can send quantum description systems at rate Q and they share maximally entangled pairs of qubits at a rate E.

Theorem 11 (State Redistribution [17]). The optimal rates for state redistribution of $|\psi_{AGBR}\rangle$ with rate-limited entanglement are

$$Q \geq \frac{1}{2} I(B; R|G)_{\psi}\,, \tag{131}$$
$$Q + E \geq H(B|G)_{\psi}\,. \tag{132}$$

We go back to the coordination setting for the cascade network (see Figure 13). Alice, Bob, and Charlie would like to simulate the joint state $|\omega_{RABC}\rangle$, where system R is a reference system owned by Alice, and purifies the state ω_{ABC}. Suppose that Alice prepares the desired state $|\omega_{\bar{R}A\bar{B}\bar{C}}\rangle^{\otimes n}$ locally in her lab, where $\bar{B}^n$, $\bar{C}^n$, and $\bar{R}^n$ are her ancillas. Let $\varepsilon > 0$ be arbitrarily small. By the state redistribution theorem, Theorem 11, Alice can transmit $\bar{B}^n \bar{C}^n$ to Bob at communication rate Q_1 and entanglement rate E_1, provided that

$$Q_1 > \frac{1}{2} I(\bar{B}\bar{C}; \bar{R})_{\omega} = \frac{1}{2} I(BC; R)_{\omega}\,, \tag{133}$$
$$Q_1 + E_1 > H(\bar{B}\bar{C})_{\omega} = H(BC)_{\omega} \tag{134}$$

(see [17]). That is, there exist a bipartite state $\Psi_{T_A T_B'}$ and encoding maps, $\mathcal{E}^{(1)}_{\bar{B}^n \bar{C}^n T_A \to M_1}$ and $\mathcal{F}^{(1)}_{M_1 T_B' \to B^n \widetilde{C}^n}$, such that

$$\left\| \tau^{(1)}_{\bar{R}^n A^n B^n \widetilde{C}^n} - \omega^{\otimes n}_{RABC} \right\|_1 \leq \varepsilon, \tag{135}$$

for a sufficiently large n, where

$$\tau^{(1)}_{\bar{R}^n A^n B^n \widetilde{C}^n} = \left[\mathrm{id}_{\bar{R}^n A^n} \otimes \mathcal{F}^{(1)}_{M_1 T_B' \to B^n \widetilde{C}^n} \circ \left(\mathcal{E}^{(1)}_{\bar{B}^n \bar{C}^n T_A \to M_1} \otimes \mathrm{id}_{T_B'} \right) \right] \left(\omega^{\otimes n}_{\bar{R}A\bar{B}\bar{C}} \otimes \Psi_{T_A T_B'} \right)\,. \tag{136}$$

Similarly, $\tilde{C}^n$ can be compressed and transmitted with rates

$$Q_2 > \frac{1}{2} I(\bar{C}; A\bar{R})_{\omega} = \frac{1}{2} I(C; AR)_{\omega}\,, \tag{137}$$
$$Q_2 + E_2 > H(\bar{C})_{\omega} = H(C)_{\omega}\,, \tag{138}$$

by Theorem 11. Namely, there exists a bipartite state $\Theta_{T_B'' T_C}$ and encoding maps, $\mathcal{F}^{(2)}_{\bar{C}^n T_B'' \to M_2}$ and $\mathcal{D}^{(2)}_{M_2 T_C \to C^n}$, such that

$$\left\| \tau^{(2)}_{\bar{R}^n A^n \bar{B}^n C^n} - \omega^{\otimes n}_{RABC} \right\|_1 \leq \varepsilon, \tag{139}$$

where

$$\tau^{(2)}_{\bar{R}^n A^n \bar{B}^n C^n} = \left[\left(\mathrm{id}_{\bar{R}^n A^n \bar{B}^n} \otimes \mathcal{D}^{(2)}_{M_2 T_C \to C^n} \right) \circ \left(\mathcal{F}^{(2)}_{\widetilde{C}^n T''_B \to M_2} \otimes \mathrm{id}_{T_C} \right) \right] (\omega^{\otimes n}_{\bar{R} A \bar{B} \bar{C}} \otimes \Theta_{T''_B T_C}) \ . \tag{140}$$

The coding operations for the cascade network are described below.

Encoding:

A) Alice prepares $|\omega_{\bar{R}A\bar{B}\bar{C}}\rangle^{\otimes n}$ locally. She applies $\mathrm{id}_{\bar{R}^n A^n} \otimes \mathcal{E}^{(1)}_{\bar{B}^n \bar{C}^n T_A \to M_1}$, and sends M_1 to Bob.

B) As Bob receives M_1, he applies

$$\mathcal{F}_{M_1 T'_B T''_B \to B^n M_2} \equiv \left(\mathrm{id}_{B^n} \otimes \mathcal{F}^{(2)}_{\widetilde{C}^n T''_B \to M_2} \right) \circ \mathcal{F}^{(1)}_{M_1 T'_B \to B^n \widetilde{C}^n} \ . \tag{141}$$

C) Charlie receives M_2 from Bob and applies $\mathcal{D}^{(2)}_{M_2 T_C \to C^n}$.

Error analysis: We trace out the reference system R and write the analysis with respect to the reduced states. The joint state after Alice's encoding is

$$\rho^{(1)}_{R^n A^n M_1 T'_B} = \left[\mathrm{id}_{R^n A^n} \otimes \mathcal{E}^{(1)}_{\bar{B}^n \bar{C}^n T_A \to M_1} \otimes \mathrm{id}_{T'_B} \right] (\omega^{\otimes n}_{RA\bar{B}\bar{C}} \otimes \Psi_{T_A T'_B}) \, . \tag{142}$$

After Bob applies his encoder, this results in

$$\begin{aligned} \rho^{(2)}_{R^n A^n B^n M_2 T_C} &= \left[\left(\mathrm{id}_{R^n A^n B^n} \otimes \mathcal{F}^{(2)}_{\widetilde{C}^n T''_B \to M_2} \otimes \mathrm{id}_{T_C} \right) \circ \left(\mathrm{id}_{R^n A^n} \otimes \mathcal{F}^{(1)}_{M_1 T'_B \to B^n \widetilde{C}^n} \otimes \mathrm{id}_{T''_B T_C} \right) \right] (\rho^{(1)}_{R^n A^n M_1 T'_B} \otimes \Theta_{T''_B T_C}) \\ &= \left(\mathrm{id}_{R^n A^n B^n} \otimes \mathcal{F}^{(2)}_{\widetilde{C}^n T''_B \to M_2} \otimes \mathrm{id}_{T_C} \right) (\tau^{(1)}_{R^n A^n B^n \widetilde{C}^n} \otimes \Theta_{T''_B T_C}) \end{aligned} \tag{143}$$

by (141), and based on the definition of $\tau^{(1)}$ in (136). According to (135), $\tau^{(1)}$ and $\omega^{\otimes n}$ are close in trace distance. By trace monotonicity under quantum channels, we have

$$\left\| \rho^{(2)}_{R^n A^n B^n M_2 T_C} - \left(\mathrm{id}_{R^n A^n B^n} \otimes \mathcal{F}^{(2)}_{\widetilde{C}^n T''_B \to M_2} \otimes \mathrm{id}_{T_C} \right) (\omega^{\otimes n}_{RAB\widetilde{C}} \otimes \Theta_{T''_B T_C}) \right\|_1 \leq \varepsilon. \tag{144}$$

As Charlie receives M_2 and encodes, the final state, at the output of the cascade network, is given by

$$\widehat{\rho}_{R^n A^n B^n C^n} = \left[\mathrm{id}_{R^n A^n B^n} \otimes \mathcal{D}^{(2)}_{M_2 T_C \to C^n} \right] (\rho^{(2)}_{R^n A^n B^n M_2 T_C}) \, . \tag{145}$$

Once more, by trace monotonicity,

$$\left\| \widehat{\rho}_{R^n A^n B^n C^n} - \tau^{(2)}_{R^n A^n B^n C^n} \right\|_1 \leq \varepsilon \, . \tag{146}$$

(see (139) and (140)). Thus, using (139), (146), and the triangle inequality, we have

$$\begin{aligned} \left\| \widehat{\rho}_{R^n A^n B^n C^n} - \omega^{\otimes n}_{RABC} \right\|_1 &\leq \left\| \tau^{(2)}_{R^n A^n B^n C^n} - \omega^{\otimes n}_{RABC} \right\|_1 + \left\| \widehat{\rho}_{R^n A^n B^n C^n} - \tau^{(2)}_{R^n A^n B^n C^n} \right\|_1 \\ &\leq 2\varepsilon \, . \end{aligned} \tag{147}$$

This completes the achievability proof for the cascade network.

B. Converse proof

We now prove the converse part for Theorem 6. Recall that in the cascade network, each party shares entanglement with their nearest neighbor a priori, i.e., Alice and Bob share $\left|\Psi_{T_A T'_B}\right\rangle$, while Bob and Charlie share $\left|\Theta_{T''_B T_C}\right\rangle$ (see Figure 13 in subsection VI-A). Alice applies an encoding map $\mathcal{E}_{\bar{A}^n T_A \to A^n M_1}$ on her part, and sends the output M_1 to Bob. As Bob receives M_1, he encodes using a map $\mathcal{F}_{M_1 T'_B T''_B \to B^n M_2}$, and sends M_2. As Charlie receives M_2, he applies an encoding channel $\mathcal{D}_{M_2 T_C \to C^n}$. Suppose that Alice prepares the state $|\omega_{R\bar{A}\bar{B}\bar{C}}\rangle^{\otimes n}$ locally, and then encodes as explained above. The protocol can be described through the following relations:

$$\rho^{(1)}_{R^n A^n M_1 T'_B} = (\mathrm{id}_{R^n} \otimes \mathcal{E}_{\bar{A}^n T_A \to A^n M_1} \otimes \mathrm{id}_{T'_B})(\omega^{\otimes n}_{R\bar{A}} \otimes \Psi_{T_A T'_B}) \, , \tag{148}$$

$$\rho^{(2)}_{R^n A^n B^n M_2 T_C} = (\mathrm{id}_{R^n A^n} \otimes \mathcal{F}_{M_1 T'_B T''_B \to B^n M_2} \otimes \mathrm{id}_{T_C})(\rho^{(1)}_{R^n A^n M_1 T'_B} \otimes \Theta_{T''_B T_C}) \, , \tag{149}$$

$$\widehat{\rho}_{R^n A^n B^n C^n} = (\mathrm{id}_{R^n A^n B^n} \otimes \mathcal{D}_{M_2 T_C \to C^n}) \left(\rho^{(2)}_{R^n A^n B^n M_2 T_C} \right) \, . \tag{150}$$

Let (Q_1, Q_2, E_1, E_2) be an achievable rate tuple for coordination in the cascade network with respect to $|\omega_{RABC}\rangle$. Then, there exists a sequence of codes such that

$$\left\| \widehat{\rho}_{R^n A^n B^n C^n} - \omega^{\otimes n}_{RABC} \right\|_1 \leq \varepsilon_n \tag{151}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Consider Alice's communication and entanglement rates, Q_1 and E_1. At this point, we may view

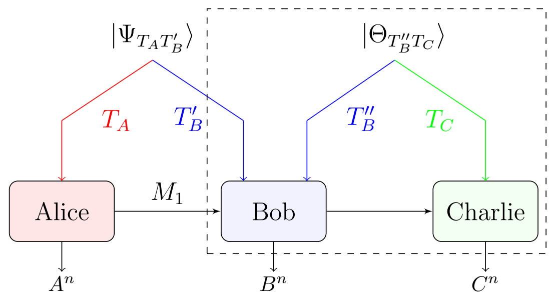


Fig. 20. At first, we treat the encoding operation of Bob and Charlie as a black box.

the entire encoding operation of Bob and Charlie as a "black box" whose input and output are (M_1, T_B') and (B^n, C^n), respectively, as illustrated in Figure 20. Now,

$$\begin{aligned} 2n(Q_1+E_1) &= 2\left[\log\dim(\mathcal{H}_{M_1}) + \log\dim(\mathcal{H}_{T_B'})\right] \\ &\geq I(M_1 T_B'; A^n R^n)_{\rho^{(1)}} \end{aligned} \tag{152}$$

since the quantum mutual information satisfies $I(A;B)_\rho \leq 2\log\dim(\mathcal{H}_A)$ in general. Therefore, by the data processing inequality,

$$\begin{aligned} I(M_1 T_B'; A^n R^n)_{\rho^{(1)}} &\geq I(B^n C^n; A^n R^n)_{\widehat{\rho}} \\ &\geq I(B^n C^n; A^n R^n)_{\omega^{\otimes n}} - n\alpha_n \\ &= n[I(BC; AR)_\omega - \alpha_n]\,, \end{aligned} \tag{153}$$

where $\alpha_n \to 0$ when $n \to \infty$. The second inequality follows from (151) and the Alicki-Fannes-Winter (AFW) inequality [110] (entropy continuity). Since $|\omega_{RABC}\rangle$ is pure, we have $I(BC;AR)_\omega = 2H(BC)_\omega$ [69, Th. 11.2.1]. Therefore, combining (152)-(153), we have

$$Q_1 + E_1 \geq H(BC)_\omega - \frac{1}{2}\alpha_n\,. \tag{154}$$

To show the bound on Q_1, observe that a lower bound on the communication rate with unlimited entanglement resources also holds with limited resources. Therefore, the bound $Q_1 \geq \frac{1}{2}I(BC;R)_\omega$ follows from the entanglement-assisted capacity theorem due to Bennett et al. [44]. It is easier to see this through resource inequalities, following the arguments in [48]. If the entanglement resources are unlimited, then the coordination code achieves

$$\begin{aligned} Q_1\,[q\to q]_{A\to BC} &\geq \langle \omega_{RBC}\rangle \\ &\equiv \langle \mathrm{Tr}_A : \omega_{RABC}\rangle \\ &\geq \frac{1}{2}I(BC;R)_\omega\,[q\to q]_{A\to BC} \end{aligned} \tag{155}$$

where the resource units $[q\to q]$, $[qq]$, and $\langle\omega_{RBC}\rangle$ represent a single use of a noiseless qubit channel, an EPR pair, and the desired state $\omega_{RBC} = \mathrm{Tr}_A(\omega_{RABC})$, respectively, while the unit resource $\langle \mathcal{N}_{A\to B} : \rho\rangle$ indicates a simulation of the channel output from $\mathcal{N}_{A\to B}$ with respect to the input state ρ. The last inequality holds by [44, 48].

Similarly, we bound Bob's communication and entanglement rates as follows,

$$2n(Q_2+E_2) = 2\left[\log\dim(\mathcal{H}_{M_2}) + \log\dim(\mathcal{H}_{T_B''})\right] \tag{156}$$
$$\geq I(M_2 T_C; A^n B^n R^n)_{\rho^{(2)}} \tag{157}$$
$$\geq I(C^n; A^n B^n R^n T_B'')_{\widehat{\rho}} \tag{158}$$
$$\geq n[I(C;ABR)_\omega - \beta_n] \tag{159}$$
$$= n[2H(C)_\omega - \beta_n] \tag{160}$$

where $\beta_n \to 0$ when $n\to\infty$. As before, the last inequality follows from (151) and the AFW inequality [110]. Hence,

$$Q_2 + E_2 \geq H(C)_\omega - \frac{1}{2}\beta_n\,. \tag{161}$$

Furthermore,

$$Q_2\,[q\to q]_{B\to C} \geq \langle\omega_{RAC}\rangle$$

$$\equiv \langle \mathrm{Tr}_B : \omega_{RABC} \rangle$$
$$\geq \frac{1}{2} I(C;AR)_\omega \, [q \to q]_{B\to C} \tag{162}$$

which implies $Q_2 \geq \frac{1}{2} I(C;AR)_\omega$.

This completes the proof of Theorem 6 for the cascade network.

XII. Broadcast Analysis (Quantum Links)

We prove the rate characterization in Theorem 8. Consider the broadcast network in Figure 15. We show achievability by using a quantum version of the binning technique.

Let $\varepsilon_i, \delta > 0$ be arbitrarily small. Define the average states,

$$\sigma_{AB}^{(x)} = \sum_{y\in\mathcal{Y}} p_{Y|X}(y|x) \omega_{AB}^{(x,y)} , \tag{163}$$
$$\sigma_{AC}^{(y)} = \sum_{x\in\mathcal{X}} p_{X|Y}(x|y) \omega_{AC}^{(x,y)} , \tag{164}$$

and consider a spectral decomposition of the reduced states of Bob and Charlie,

$$\sigma_{B}^{(x)} = \sum_{z\in\mathcal{Z}} p_{Z|X}(z|x) \, |\psi_{x,z}\rangle\langle\psi_{x,z}| \, , \tag{165}$$
$$\sigma_{C}^{(y)} = \sum_{w\in\mathcal{W}} p_{W|Y}(w|y) \, |\phi_{y,w}\rangle\langle\phi_{y,w}| \, , \tag{166}$$

where $p_{Z|X}$ and $p_{W|Y}$ are conditional probability distributions, and $\{|\psi_{x,z}\rangle\}_z, \{|\phi_{y,w}\rangle\}_w$ are orthonormal bases for $\mathcal{H}_B$, $\mathcal{H}_C$, respectively, for every given $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. We can also assume that the different bases are orthogonal to each other by requiring that Bob and Charlie encode on a different Hilbert space for every value of (x, y).

We use the type class definitions and notations in [69, Chap. 14]. In particular, $T_\delta^{X^n}$ denotes the δ-typical set with respect to p_X, and $T_\delta^{Z^n|x^n}$ is the conditional δ-typical set with respect to p_{XZ}, given $x^n \in T_\delta^{X^n}$.

Classical Codebook Generation: For every sequence $z^n \in \mathcal{Z}^n$, assign an index $m_1(z^n)$, uniformly at random from $[2^{nQ_1}]$. A bin $\mathfrak{B}_1(m_1)$ is defined as the subset of sequences in $\mathcal{Z}^n$ that are assigned the same index m_1, for $m_1 \in [2^{nQ_1}]$. The codebook is revealed to all parties.

Encoding:

A) Alice prepares $\omega_{A\bar{B}\bar{C}}^{\otimes n}$ locally, where $\bar{B}^n\bar{C}^n$ are her ancillas, without any correlation with X^n and Y^n (see Remark 6). She applies the encoding channel $\mathcal{E}_{\bar{B}^n\to M_1}^{(1)} \otimes \mathcal{E}_{\bar{C}^n\to M_2}^{(2)}$,

$$\mathcal{E}_{\bar{B}^n\to M_1}^{(1)}(\rho_1) = \sum_{x^n\in\mathcal{X}^n} p_X^{\otimes n}(x^n) \sum_{z^n\in\mathcal{Z}^n} \langle\psi_{x^n,z^n}| \, \rho_1 \, |\psi_{x^n,z^n}\rangle \, |m_1(z^n)\rangle\langle m_1(z^n)| \, , \tag{167}$$
$$\mathcal{E}_{\bar{C}^n\to M_2}^{(2)}(\rho_2) = \sum_{y^n\in\mathcal{Y}^n} p_Y^{\otimes n}(y^n) \sum_{w^n\in\mathcal{W}^n} \langle\phi_{y^n,w^n}| \, \rho_2 \, |\phi_{y^n,w^n}\rangle \, |m_2(w^n)\rangle\langle m_2(w^n)| \, , \tag{168}$$

for $\rho_1 \in \Delta(\mathcal{H}_B^{\otimes n})$, $\rho_2 \in \Delta(\mathcal{H}_C^{\otimes n})$, and transmits M_1 and M_2 to Bob and Charlie, respectively.

B) First, Bob applies the following encoding channel,

$$\mathcal{F}_{M_1\to B^n}^{(x^n)}(\rho_{M_1}) = \sum_{m_1=1}^{2^{nQ_1}} \langle m_1| \, \rho_{M_1} \, |m_1\rangle \left(\frac{1}{\left|T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1)\right|} \sum_{z^n\in T_\delta^{Z^n|x^n}\cap\mathfrak{B}_1(m_1)} |\psi_{x^n,z^n}\rangle\langle\psi_{x^n,z^n}| \right) \tag{169}$$

C) Charlie encodes in a similar manner.

Error analysis: Due to the code construction, it suffices to consider the individual errors of Bob and Charlie,

$$\frac{1}{2} \left\| \omega_{XAB}^{\otimes n} - \left(\mathcal{F}_{X^nM_1\to X^nB^n} \circ \mathcal{E}_{\bar{B}^n\to M_1}^{(1)} \right) (\omega_X^{\otimes n} \otimes \omega_{A\bar{B}}^{\otimes n}) \right\|_1 , \tag{170}$$
$$\frac{1}{2} \left\| \omega_{YAC}^{\otimes n} - \left(\mathcal{D}_{Y^nM_2\to Y^nC^n} \circ \mathcal{E}_{\bar{C}^n\to M_2}^{(2)} \right) (\omega_Y^{\otimes n} \otimes \omega_{A\bar{C}}^{\otimes n}) \right\|_1 , \tag{171}$$

respectively, where we use the short notation $\mathcal{E}_{\bar{B}^n\to M_1}^{(1)} \equiv \mathrm{id}_{X^nA^n} \otimes \mathcal{E}_{\bar{B}^n\to M_1}^{(1)}$, and similarly for the other encoding maps.

We now focus on Bob's error. Consider a given codebook $\mathscr{C}_1 = \{m_1(z^n)\}$. Alice encodes M_1 by

$$\mathcal{E}_{\bar{B}^n\to M_1}^{(1)}(\omega_{AB}^{\otimes n}) = \sum_{\tilde{x}^n\in\mathcal{X}^n} p_X^{\otimes n}(\tilde{x}^n) \sum_{z^n\in\mathcal{Z}^n} \langle\psi_{\tilde{x}^n,z^n}| \, \omega_{AB}^{\otimes n} \, |\psi_{\tilde{x}^n,z^n}\rangle \, |m_1(z^n)\rangle\langle m_1(z^n)| \, , \tag{172}$$

where we use the short notation $|\psi\rangle_{x^n,z^n} \equiv \bigotimes_{i=1}^n |\psi\rangle_{x_i,z_i}$. By the weak law of large numbers, this state is ε_1-close in trace distance to

$$\begin{aligned}\rho^{(1)}_{A^n M_1} &= \sum_{\tilde{x}^n \in T_\delta^{X^n}} p_X^{\otimes n}(\tilde{x}^n) \sum_{z^n \in T_\delta^{Z^n|\tilde{x}^n}} \langle \psi_{\tilde{x}^n,z^n} | \, \sigma^{(\tilde{x}^n)}_{A^n \bar{B}^n} \, |\psi_{\tilde{x}^n,z^n}\rangle \, |m_1(z^n)\rangle\!\langle m_1(z^n)| \\ &= \sum_{x^n \in T_\delta^{X^n}} p_X^{\otimes n}(x^n) \rho^{(1|x^n)}_{A^n M_1} ,\end{aligned} \tag{173}$$

for a sufficiently large n, where we have defined

$$\rho^{(1|x^n)}_{A^n M_1} = \sum_{z^n \in T_\delta^{Z^n|x^n}} \langle \psi_{x^n,z^n} | \, \sigma^{(x^n)}_{A^n \bar{B}^n} \, |\psi_{x^n,z^n}\rangle \, |m_1(z^n)\rangle\!\langle m_1(z^n)| \; . \tag{174}$$

Let $x^n \in T_\delta^{X^n}$. After Bob encodes B^n, we have

$$\mathcal{F}^{(x^n)}_{M_1 \to B^n}\left(\rho^{(1|x^n)}_{A^n M_1}\right) = \sum_{z^n \in T_\delta^{Z^n|x^n}} \langle \psi_{x^n,z^n} | \, \sigma^{(x^n)}_{A^n \bar{B}^n} \, |\psi_{x^n,z^n}\rangle \, \mathcal{F}^{(x^n)}_{M_1 \to B^n}(|m_1(z^n)\rangle\!\langle m_1(z^n)|) \, . \tag{175}$$

By the definition of Bob's encoding channel, $\mathcal{F}^{(x^n)}_{M_1 \to B^n}$, in (169),

$$\mathcal{F}^{(x^n)}_{M_1 \to B^n}(|m_1(z^n)\rangle\!\langle m_1(z^n)|) = \frac{1}{\left|T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n))\right|} \sum_{\tilde{z}^n \in T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n))} |\psi_{x^n,\tilde{z}}\rangle\!\langle \psi_{x^n,\tilde{z}}| \; . \tag{176}$$

Substituting in (175) yields

$$\begin{aligned}\mathcal{F}^{(x^n)}_{M_1 \to B^n}\left(\rho^{(1|x^n)}_{A^n M_1}\right) = &\sum_{z^n \in T_\delta^{Z^n|x^n}} \langle \psi_{x^n,z^n} | \, \sigma^{(x^n)}_{A^n \bar{B}^n} \, |\psi_{x^n,z^n}\rangle \\ &\otimes \left[\frac{1}{\left|T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n))\right|} \sum_{\tilde{z}^n \in T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n))} |\psi_{x^n,\tilde{z}^n}\rangle\!\langle \psi_{x^n,\tilde{z}^n}| \right] .\end{aligned} \tag{177}$$

Based on the classical result [112, Chapter 10.3], the random codebook $\mathscr{C}_1$ satisfies that

$$\Pr_{\mathscr{C}_1}\left(\exists \tilde{z}^n \in T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n)) : \tilde{z}^n \neq z^n\right) \leq \varepsilon_2 \tag{178}$$

given $z^n \in T_\delta^{Z^n|x^n}$, for a sufficiently large n, provided that the codebook size is at least $2^{n(H(Z|X)+\varepsilon_3)}$, where $H(Z|X)$ denotes the classical conditional entropy. As $|\mathscr{C}_1| = 2^{nQ_1}$, this holds if

$$\begin{aligned}Q_1 &> H(Z|X) + \varepsilon_3 \\ &= H(B|X)_\omega + \varepsilon_3 \, .\end{aligned} \tag{179}$$

Observe that if the summation set in (177), $T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n))$, consists of the sequence z^n alone, then the overall state in (177) is identical to the post-measurement state after a typical subspace measurement on B^n, with respect to the conditional δ-typical set $T_\delta^{Z^n|x^n}$. Based on the gentle measurement lemma [113], this state is ε_4-close to $\sigma^{(x^n)}_{AB}$, for a sufficiently large n.

Therefore, by the triangle inequality and total expectation formula,

$$\begin{aligned}&\left\| \omega^{\otimes n}_{XAB} - \mathbb{E}_{\mathscr{C}_1}\left(\mathcal{F}_{X^n M_1 \to X^n B^n} \circ \mathcal{E}^{(1)}_{\bar{B}^n \to M_1}\right)\left(\omega_X^{\otimes n} \otimes \omega_{A\bar{B}}^{\otimes n}\right) \right\|_1 \\ &\leq \sum_{x^n \in \mathcal{X}^n} p_X^{\otimes n}(x^n) \cdot \mathbb{E}_{\mathscr{C}_1} \left\| \sigma^{(x^n)}_{A^n B^n} - \left(\mathcal{F}^{(x^n)}_{M_1 \to B^n} \circ \mathcal{E}^{(1)}_{\bar{B}^n \to M_1}\right)\left(\sigma^{(x^n)}_{A^n B^n}\right) \right\|_1 \\ &\leq \varepsilon_1 + \varepsilon_2 + \varepsilon_4 \, .\end{aligned} \tag{180}$$

By symmetry, Charlie's error tends to zero as well, provided that $Q_2 \geq H(C|Y)_\omega + \varepsilon_5$. Since the total error vanishes, when averaged over the class of binning codebooks, it follows that there exists a deterministic codebook with the same property. The achievability proof follows by taking $n \to \infty$ and then $\varepsilon_j, \delta \to 0$.

The converse proof follows the lines of [17], and it is thus omitted. This completes the proof of Theorem 8 for the broadcast network.

XIII. Multiple-Access Analysis (Quantum Links)

We prove the rate characterization in Theorem 9. Consider the multiple-access network in Figure 16. As explained in Remark 8, coordination in the multiple-access network is only possible if there exists an isometry $V : \mathcal{H}_C \to \mathcal{H}_{C_1} \otimes \mathcal{H}_{C_2}$ such that

$$(\mathbb{1} \otimes V) |\omega_{ABC}\rangle = |\phi_{AC_1}\rangle \otimes |\chi_{BC_2}\rangle \tag{181}$$

where $|\phi_{AC_1}\rangle$ and $|\chi_{BC_2}\rangle$ are purifications of ω_A and ω_B, respectively. For this reason, Theorem 9 assumes that this property holds. Furthermore, since $|\phi_{AC_1}\rangle$ and $|\chi_{BC_2}\rangle$ purify ω_A and ω_B, respectively, we have $H(C_1)_\phi = H(A)_\phi = H(A)_\omega$ and $H(C_2)_\chi = H(B)_\chi = H(B)_\omega$. Thus, it suffices to show that (Q_1, Q_2) is achievable if and only if

$$Q_1 \geq H(C_1)_\phi \,, \tag{182}$$
$$Q_2 \geq H(C_2)_\chi \,. \tag{183}$$

The achievability proof follows from the Schumacher compression protocol [47] [69, chap. 18] in a straightforward manner. Alice and Bob prepare $\phi_{AC_1}^{\otimes n}$ and $\chi_{BC_2}^{\otimes n}$, respectively. Then, they send C_1^n and C_2^n using the Schumacher compression protocol, and finally, Charlie applies the isometry $(V^\dagger)^{\otimes n}$ in order to simulate $\omega_{ABC}^{\otimes n}$ (see (181)). The details are omitted.

It remains to show the converse part. Recall that in the multiple-access network, Alice and Bob each applies their respective encoding map, $\mathcal{E}_{A^n \to A^n M_1}$ and $\mathcal{F}_{B^n \to B^n M_2}$, and send the quantum descriptions M_1 and M_2. Then, Charlie encodes by $\mathcal{D}_{M_1 M_2 \to C^n}$.

The protocol can be described through the following relations:

$$\rho^{(1)}_{A^n M_1} = \mathcal{E}_{A^n \to A^n M_1}(\omega_A^{\otimes n}) \,, \; \rho^{(2)}_{B^n M_2} = \mathcal{F}_{B^n \to B^n M_2}(\omega_B^{\otimes n}) \,, \tag{184}$$
$$\widehat{\rho}_{A^n B^n C^n} = (\mathrm{id}_{A^n B^n} \otimes \mathcal{D}_{M_1 M_2 \to C^n}) \, (\rho^{(1)}_{A^n M_1} \otimes \rho^{(2)}_{B^n M_2}) \,. \tag{185}$$

Let (Q_1, Q_2) be an achievable rate pair for coordination in the multiple-access network in Figure 16. Then, there exists a sequence of $(2^{nQ_1}, 2^{nQ_2}, n)$ coordination codes such that

$$\left\| \widehat{\rho}_{A^n B^n C^n} - \omega_{ABC}^{\otimes n} \right\|_1 \leq \varepsilon_n \tag{186}$$

tends to zero as $n \to \infty$. Applying the isometry $V^{\otimes n}$ yields

$$\left\| \widehat{\sigma}_{A^n B^n C_1^n C_2^n} - \phi_{AC_1}^{\otimes n} \otimes \chi_{BC_2}^{\otimes n} \right\|_1 \leq \varepsilon_n \,, \tag{187}$$

by (181), where

$$\widehat{\sigma}_{A^n B^n C_1^n C_2^n} = (\mathbb{1}_{AB} \otimes V)^{\otimes n} \widehat{\rho}_{A^n B^n C^n} (\mathbb{1}_{AB} \otimes V^\dagger)^{\otimes n} \,. \tag{188}$$

It thus follows that

$$\left\| \widehat{\sigma}_{A^n C_1^n} - \phi_{AC_1}^{\otimes n} \right\|_1 \leq \varepsilon_n \tag{189}$$

and

$$\left\| \widehat{\sigma}_{B^n C_2^n} - \chi_{BC_2}^{\otimes n} \right\|_1 \leq \varepsilon_n \,. \tag{190}$$

Now, Alice's communication rate is bounded by

$$\begin{aligned} 2nQ_1 &\overset{(a)}{\geq} I(M_1; A^n | M_2)_{\rho^{(1)} \otimes \rho^{(2)}} \\ &\overset{(b)}{=} I(M_1 M_2; A^n)_{\rho^{(1)} \otimes \rho^{(2)}} \\ &\overset{(c)}{\geq} I(C^n; A^n)_{\widehat{\rho}} \\ &\overset{(d)}{=} I(C_1^n C_2^n; A^n)_{\widehat{\sigma}} \\ &\overset{(e)}{\geq} I(C_1^n C_2^n; A^n)_\omega - n\alpha_n \\ &\overset{(f)}{=} 2nH(C_1)_\phi - n\alpha_n \\ &\overset{(g)}{=} 2nH(A)_\omega - n\alpha_n \,, \end{aligned} \tag{191}$$

where (a) holds because M_1 is of dimension 2^{nQ_1}, (b) since $I(M_2; A^n)_{\rho^{(1)} \otimes \rho^{(2)}} = 0$, (c) follows from the data processing inequality, (d) holds since the von Neumann entropy is isometrically invariant, (e) by the AFW inequality [110] , (f) since the mutual information is calculated with respect to the product state $|\phi_{AC_1}\rangle^{\otimes n} \otimes |\chi_{BC_2}\rangle^{\otimes n}$, and (g) holds since $|\phi_{AC_1}\rangle$ is a purification of ω_A. The bound on Bob's communication rate follows by symmetry. This completes the proof of Theorem 9. $\square$

XIV. Summary and discussion

A. Summary

We study coordination in three network models with classical communication links: 1) two-node network simulating a classical-quantum (c-q) state, 2) no-communication network simulating a separable state, and 3) a broadcast network simulating a classical-quantum-quantum (c-q-q) state, and consider coordination in additional three networks with quantum links: 1) a cascade network simulating quantum states with limited communication and entanglement assistance, 2) a quantum linked broadcast network simulating a joint quantum state with classical side information, and 3) a multiple-access network, generating entanglement between each sender and the receiver. We observe that the network topology dictates the type of states that can be simulated. Our findings generalize classical results from [45] and [99], and also quantum results from [10].

The results are relevant for various applications, where the network nodes could represent classical-quantum sensors [114], computers performing a joint computation task [115, 116], or players in a nonlocal game [117, 118] as we illustrated in the broadcast network with quantum links, in which we establish the optimal rates required to achieve a certain quantum correlation to win a game at a desired probability.

B. General Coordination Problem

The coordination problem in its general form can be presented as follows. Given a network consisting of N nodes, the nodes need to cooperate using limited resources to asymptotically achieve a joint state that is arbitrarily close in trace distance to a desired joint state $\omega_{A_1^n \dots A_N^n}$. The objective of the coordination task is to find the optimal resources needed for simulating the correlation manifested in the desired joint state. We represent each user in the coordination network by a node, each node is assigned an index $j \in [N]$ and has access to a quantum system A_j^n. The users may communicate using classical or quantum communication links. In this paper we considered one-way links, in general they can be two-way links. We denote the classical message transmitted from Node j to Node k by $m_{j,k}$ with a rate $R_{j,k}$, similarly, a quantum message (description) is denoted by $M_{j,k}$, transmitted at a rate $Q_{j,k}$. In addition to the limited communication links, the users have access to additional resources such as common randomness (CR) at a rate R_0, entanglement assistance at a rate $E_{j,k}$, and side information for Node j which can be either classical or quantum. Each user performs an encoding operation $\mathcal{E}_j$ on his system A_j, where he applies the encoding operation on the resources available in his possession. Receiving side information, incoming classical messages $m_{l,j}$ and quantum descriptions $M_{l,j}$, using his share of CR and entanglement assistance, he prepares the state of his system ρ_{A_j}, in addition to the outgoing message $m_{j,k}$, and the joint state of the quantum descriptions $M_{j,k}$. After n time steps, the network ends up in a joint state $\widehat{\rho}_{R^n A_1^n \dots A_N^n}$, where R^n is a reference system. The users would like this state to be close to $\omega_{RA_1 \dots A_N}^{\otimes n}$. Below, we provide generic definitions for the coordination problem.

Definition 9. A rate tuple $(R_0, \{R_{j,k}\}, \{E_{j,k}\}, \{Q_{j,k}\})_{(j,k)\in[N]}$ is achievable for the simulation of $\omega_{A_1 \dots A_N}$ in the network if for every $\varepsilon, \delta > 0$ and a sufficiently large n, there exists a $\left(2^{nR_0}, \left\{2^{nR_{j,k}}\right\}, \left\{2^{nE_{j,k}}\right\}, \left\{2^{nQ_{j,k}}\right\}, n\right)_{j,k\in[N]}$ code that achieves

$$\left\| \widehat{\rho}_{R^n A_1^n \dots A_N^n} - \omega_{RA_1 \dots A_N}^{\otimes n} \right\|_1 \leq \varepsilon \,. \tag{192}$$

Definition 10. The coordination capacity region with respect to a desired state $\omega_{A_1 \dots A_N}^{\otimes n}$, is the closure of the set of all achievable rate tuples.

This work can be viewed as a step forward in understanding coordination in a general network that may comprise either classical or quantum resources. While our results cover fundamental building blocks for quantum network coordination, we do not claim to have solved all network settings. The no-communication network was independently considered by George et al. [10] (see also [96, 97]) in both the one-shot and asymptotic settings. Another interesting direction for future work is the characterization of network coordination in the one-shot regime for the two-node, broadcast, multiple-access and cascade networks.

C. Correlation resources

We study the effect of pre-shared correlation on network coordination, focusing on pre-shared common randomness (CR) in networks with classical links, and pre-shared entanglement with quantum links. George and Cheng [76] have recently considered a broadcast setting of state splitting, including classical links and entanglement assistance (see also [9]). It would be interesting to further study coordination networks with classical links and pre-shared entanglement at a limited rate. The general task of quantum network coordination can be viewed as a generalization of channel/source simulation [6–12], resolvability and soft covering [43, 55, 92, 93], state merging [13, 15], state redistribution [17, 19], entanglement dilution [20–22, 82], randomness extraction [23, 24], source coding [25–28], and many others (see Table I).

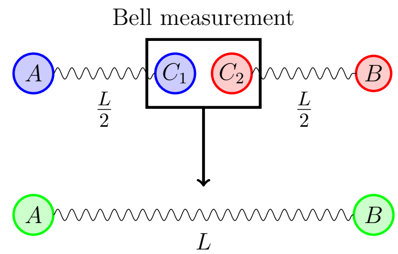


Fig. 21. A quantum repeater link [119]. Initially, systems C_1 and C_2 belong to Charlie. C_1 is maximally entangled with Alice's system A (the blue pair), while C_2 is maximally entangled with Bob's system B (the red pair). Both pairs of entangled systems are at $\frac{L}{2}$ distance apart. When Charlie performs a local Bell measurement on his systems, causing entanglement swapping. Such that at the end of the protocol, systems A and B are maximally entangled (the green pair), which are at a distance L from each other, creating a longer distance entanglement which can be exploited further .

D. Quantum repeaters

Quantum communication relies on the transmission of quantum signals over long distances [120]. Unfortunately, long-range communication is limited by attenuation and loss [121]. While classical systems can overcome such losses with straightforward signal amplification, this solution is not viable for quantum signals due to the no-cloning theorem, which prohibits the replication of quantum states. Therefore, alternative methods are required to address these limitations.

Quantum repeaters offer a promising solution by enabling the generation of entanglement between distant network nodes. The process begins by creating maximally entangled pairs between neighboring nodes, and then extending this entanglement to a longer range [122]. This long-range entanglement facilitates quantum teleportation, which enables the transmission of quantum information between the sender and receiver. Quantum repeaters are expected to play a central role in the future quantum Internet [123], and recent advancements in their implementation have been explored across various experimental platforms [124–130].

In the simplest description of a quantum repeater, the process begins with using quantum communication and entanglement distillation to prepare two pairs of qubits at maximally entangled states, namely, $|\Phi_{AC_1}\rangle$ between the sender and the repeater, and $|\Phi_{BC_2}\rangle$ between the repeater and the receiver. At the next stage, the repeater performs a Bell measurement on C_1 and C_2, thus swapping the entanglement such that A and B are now entangled at a distance twice that of the initial entangled pairs. See Figure 21. Different information-theoretic models for quantum repeaters can be found in [131–137].

Coordination is highly relevant for quantum repeaters. For instance, consider the simulation of a product of two maximally entangled pairs of qubits, $|\omega_{ABC_1C_2}\rangle = |\Phi_{AC_1}\rangle \otimes |\Phi_{BC_2}\rangle$ over the multiple-access network in Figure 16. The coordination capacity region for this setup is $\mathcal{Q}_{\text{MAC}}(\omega) = \{(Q_1, Q_2) : Q_i \geq 1\}$, which reflects the requirement of sending a single qubit from the repeater (Charlie), to each user (Alice and Bob), to prepare the two entangled pairs. See Remark 11. As described above, a quantum repeater can transform this correlation into maximal entanglement between A and B. More generally, a quantum repeater can be adapted to simulate a wide range of quantum correlations, with the required communication rates being determined by the results of Theorem 9. Thus, the coordination features of a multiple-access network are not only relevant but also beneficial in the context of quantum repeater networks. The findings from Theorem 9 highlight the optimal communication rates necessary for generating entangled states beyond just maximally entangled pairs, broadening the potential applications of quantum repeaters.

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