Quantum Coordination Rates in Multi-User Networks

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Abstract

Quantum coordination is considered in networks with classical and quantum links. We begin with networks with *classical links*, and characterize the generation of separable and classical-quantum correlations in three primary models: 1) a two-node network with limited common randomness (CR), 2) a no-communication network, and 3) a broadcast network, which consists of a single sender and two receivers. We establish the optimal tradeoff between the classical communication and CR rates in each setting, thus characterizing the minimal resources for simulating classical-quantum correlations.

Next, we consider coordination in networks with *quantum links*. We study the following models: 1) a cascade network with limited entanglement, 2) a broadcast network, and 3) a multiple-access network with two senders and a single receiver. We establish the optimal tradeoff between quantum communication and entanglement rates in each setting, characterizing the minimal resources for entanglement coordination. The examples demonstrate that coordination of entanglement and coordination of separable correlations behave differently. At last, we show the implications of our results on nonlocal games with quantum strategies.

Index Terms

Quantum communication, coordination, reverse Shannon theorem, entanglement distribution.

I. Introduction

State distribution and coordination are important in quantum communication [1], computation [2], and cryptography [3]. The quantum coordination problem can be described as follows. Consider a network that consists of N nodes, where Node i can perform an encoding operation $\mathcal{E}_i$ on a quantum system A_i, and its state should be in a certain correlation with the rest of the network nodes. The objective is to simulate a specific joint state $\omega_{A_1,A_2,\dots,A_N}$. Node i can send qubits to node j via a quantum channel at a limited rate $Q_{i,j}$. The nodes may also share limited entanglement resources, prior to their communication. The optimal performance is characterized by the quantum communication rates $Q_{i,j}$ that are necessary and sufficient for simulating the desired quantum correlation. Alternatively, the nodes may send bits using classical communication links at a limited rate $R_{i,j}$. Instances of the network coordination problem include channel/source simulation [4–9], state merging [10, 11], state redistribution [12, 13], entanglement dilution [14–17], randomness extraction [18, 19], source coding [20–23], and many others.

Two-node classical coordination: In classical coordination, the goal is to simulate a joint probability distribution. In the basic two-node network, see Figure 1, two users would like to simulate a joint distribution p_{XY}. This can be achieved if and only if the *classical* communication rate $R_{1,2}$ is above Wyner's common information [24], defined as:

$$C(X;Y) \triangleq \min I(U;XY)\,, \tag{1}$$

where the minimum is taken over all auxiliary variables U that satisfy the Markov relation $X \multimap U \multimap Y$, and $I(U;XY)$ is the mutual information between U and (X,Y). One may also consider the case where the nodes share classical correlation resources, a priori, in the form of common randomness. Given a sufficient amount of pre-shared common randomness, the desired distribution can be simulated if and only if $R_{1,2} \geq I(X;Y)$ [25, 26].

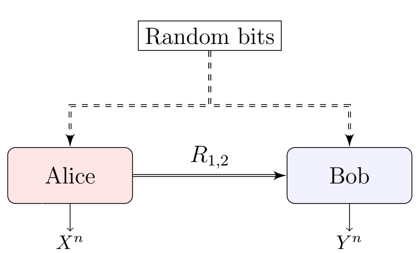


Fig. 1: Classical two-node network

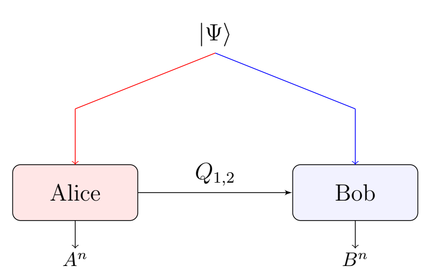


Fig. 2: Quantum two-node network

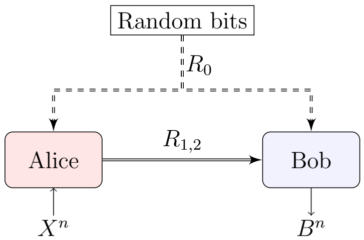


Fig. 3: Two-node network with classical links

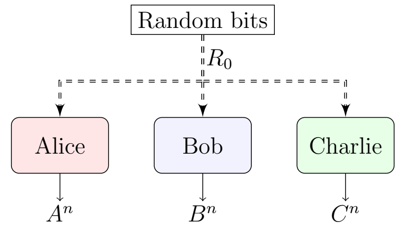


Fig. 4: No-communication network with common randomness

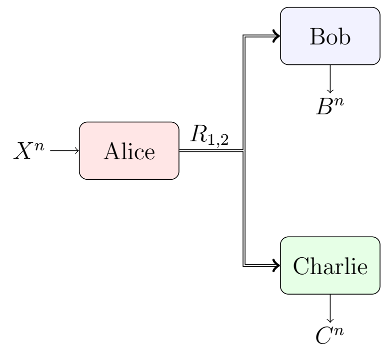


Fig. 5: Broadcast network with classical links

Two-node quantum coordination: In the quantum setting, see Figure 2, the goal is to simulate a joint state. A bipartite state ω_{AB} can be simulated if and only if the quantum communication rate is above the von Neumann entropy [15], i.e., $Q_{1,2} \geq H(\omega_B)$, with $H(\rho) = -\text{Tr}(\rho \log(\rho))$, where ω_B is the reduced state of ω_{AB}. Now, suppose that the nodes share entanglement resources, prior to their communication. Based on the quantum reverse Shannon theorem [27], given sufficient entanglement, the desired state can be simulated if and only if the quantum communication rate satisfies $Q_{1,2} \geq \frac{1}{2} I(A;B)_\omega$, where $I(A;B)_\omega$ is the quantum mutual information.

Multi-node quantum coordination: In this work, we consider quantum coordination in networks with either classical links or quantum links. The models of coordination in multi-user networks with classical links are motivated by quantum-enhanced Internet of Things (IoT) networks in which the communication links are classical [28–31], and the study of coordination in models including quantum links is motivated by applications such as the quantum Internet and quantum repeaters [32]. In each network, we determine the optimal coordination rates, characterizing the minimal resources required in order to simulate a joint quantum state among multiple parties. We further discuss the implications of our results on nonlocal quantum games. In particular, coordination in the broadcast network in Figure 7 can be viewed as a sequential game, where a coordinator (the sender) provides the players (the receivers) with quantum resources. In the course of the game, the referee sends questions, X^n and Y^n, to each player, and they respond with B^n and C^n. In order to win the game with a certain probability, the communication rates must satisfy the constraints with respect to an appropriate correlation.

Our work is divided into two parts, focusing on classical links and quantum links.

A. *Classical Links*

We first consider quantum coordination in three multi-user networks with classical communication links, where Node i sends classical bits to Node j at a limited rate $R_{i,j}$. While classical links cannot generate entanglement, we may consider the simulation of classical-quantum (c-q) and separable states in multi-user networks, where shared random bits are available to the network users at a limited rate. This resource is referred to as common randomness (CR).

We study three networks with classical links. Our results are summarized below.

1) Two-Node Network: Consider a two-user network as in Figure 3. Alice and Bob aim to simulate a c-q state $\omega_{XB}^{\otimes n}$. Before communication begins, Alice and Bob share CR in a limited bit rate R_0. Then, Alice sends classical bits at a rate $R_{1,2}$ to Bob. We characterize the optimal tradeoff between the required rate of description and the amount of CR used. Specifically, a rate pair $(R_0, R_{1,2})$ is called achievable if there exists a sequence of coordination codes such that the encoded state $\rho_{X^n B^n}$ is ε_n-close to $\omega_{XB}^{\otimes n}$, where ε_n tends to zero as $n \to \infty$. We show that coordination can be achieved iff the rate pair $(R_0, R_{1,2})$ satisfies

$$R_{1,2} \geq I(X;U)_\sigma \,, \tag{2}$$

$$R_0 + R_{1,2} \geq I(XB;U)_\sigma \,, \tag{3}$$

for some c-q extension σ_{XUB} of the form

$$\sigma_{XUB} = \sum_{(x,u)\in\mathcal{X}\times\mathcal{U}} p_{X,U}(x,u) \, |x\rangle\!\langle x|_X \otimes |u\rangle\!\langle u|_U \otimes \theta_B^u \,. \tag{4}$$

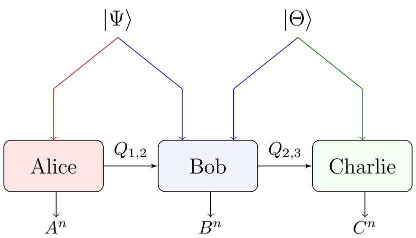


Fig. 6: Cascade network with quantum links and pre-shared entanglement

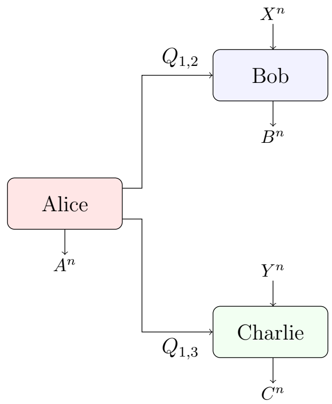


Fig. 7: Broadcast network with quantum links

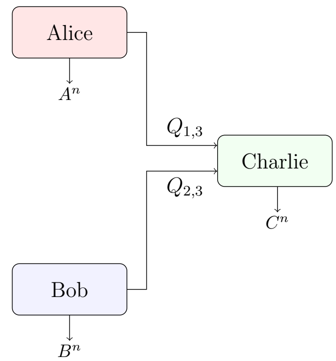


Fig. 8: Multiple access network with quantum links

2) No-Communication Network: Our second model is a no-communication network, see Figure 4, where three users, Alice, Bob, and Charlie would like to simulate a separable state ω_{ABC}, given CR at a rate R_0, and no communication. We show that the optimal CR rate is $R_0 = \inf I(U;ABC)_\sigma$, where the infimum is over the set of all extensions

$$\sigma_{UABC} = \sum_{u\in\mathcal{U}} p_U(u)\ |u\rangle\langle u|_U \otimes \theta_A^u \otimes \theta_B^u \otimes \theta_C^u \tag{5}$$

such that $\sigma_{ABC} = \omega_{ABC}$. Note that A, B and C are uncorrelated when conditioned on U.

3) Broadcast Network: In the broadcast network in Figure 5, a single sender and two receivers wish to simulate a classical-quantum-quantum state ω_{XBC}. We establish that the state ω_{XBC} can be simulated in the broadcast network in Figure 5 iff the rate pair $(R_0, R_{1,2})$ satisfies

$$R_{1,2} \geq I(X;U)_\sigma\,, \tag{6}$$
$$R_0 + R_{1,2} \geq I(XBC;U)_\sigma\,, \tag{7}$$

for an extension σ_{XUBC} that satisfies a Markov property.

B. Quantum Links

In the second part of our work, we consider coordination with quantum links, where Node i sends qubits to Node j at a limited rate $Q_{i,j}$. We study three multi-user networks of this form.

1) Cascade Network: We begin with the cascade network in Figure 6. Alice, Bob, and Charlie wish to simulate a joint quantum state ω_{ABC}. Let $|\omega_{ABCR}\rangle$ be a purification of the desired state. Before communication begins, each party shares entanglement with their nearest neighbor, at a limited rate. Now, Alice sends qubits to Bob at a rate $Q_{1,2}$, and thereafter, Bob sends qubits to Charlie at a rate $Q_{2,3}$. We show that ω_{ABC} can be simulated iff the rate tuple $(Q_{1,2}, E_{1,2}, Q_{2,3}, E_{2,3})$ satisfies

$$Q_{1,2} \geq \frac{1}{2} I(BC;R)_\omega\,, \tag{8}$$
$$Q_{1,2} + E_{1,2} \geq H(BC)_\omega\,, \tag{9}$$
$$Q_{2,3} \geq \frac{1}{2} I(C;RA)_\omega\,, \tag{10}$$
$$Q_{2,3} + E_{2,3} \geq H(C)_\omega\,, \tag{11}$$

where $E_{i,j}$ is the entanglement rate between Node i and Node j and $|\omega_{ABCR}\rangle$ is a purification of ω_{ABC}.

We provide two examples showing how the capacity behavior changes when simulating a mixture versus a tripartite entangled state (see Figure 13).

2) Quantum Broadcast Network: Next, we study the quantum broadcast network shown in Figure 7. Consider a network with a single sender, Alice, and two receivers, Bob and Charlie, where the latter are provided with classical sequences of information X^n and Y^n. We show that the state ω_{XYABC} can be simulated iff the rate pair $(Q_{1,2}, Q_{1,3})$ satisfies:

$$Q_{1,2} \geq H(B|X)_\omega\,, \tag{12}$$

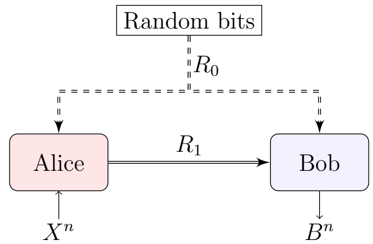


Fig. 9: Two-node network.

$$Q_{1,3} \geq H(C|Y)_{\omega}, \tag{13}$$

where ω_{XYABC} is the desired joint state.

3) Multiple-Access Network: The third quantum-link setting is the multiple-access network shown in Figure 8. In this setting we have two transmitters, Alice and Bob, and one receiver, Charlie. We observe that since there is no cooperation between the transmitters, a joint state ω_{ABC} can only be simulated if it is isometrically equivalent to a state of the form $\omega_{AC_1} \otimes \omega_{BC_2}$. We show that the state ω_{ABC} can be simulated iff the rate pair $(Q_{1,3}, Q_{2,3})$ satisfies:

$$Q_{1,3} \geq H(C_1)_{\omega}\,, \tag{14}$$

$$Q_{2,3} \geq H(C_2)_{\omega} \tag{15}$$

We further discuss the implications of our results on nonlocal quantum games. In particular, coordination in the broadcast network in Figure 7 can be viewed as a sequential game, where a coordinator (the sender) provides the players (the receivers) with quantum resources. In the course of the game, the referee sends questions, X^n and Y^n, to each player, and they respond with B^n and C^n. In order to win the game with a certain probability, the communication rates must satisfy the constraints with respect to an appropriate correlation.

In the analysis, we use different techniques for different networks, including quantum resolvability results [33–35], random coding, the state redistribution theorem [12], and the Schumacher compression protocol. In the broadcast network with quantum links, we assume that Alice does not have prior correlation with Bob and Charlie's resources X^n and Y^n. Therefore, the standard techniques of state redistribution [12] or quantum source coding with side information [36] are not suitable for our purposes. Instead, we generalize the method of classical binning [37] to handle the quantum case.

The paper is organized as follows. In Section II, we introduce the notation conventions. In Section III, we consider coordination of c-q and separable correlations in networks that consist of classical links. We present the coding definitions and results for the two-node network, the no-communication network, and the broadcast network in Subsections III-A, III-B, and III-C, respectively. In Section IV, we consider entanglement coordination in networks that consist of quantum links. We address the cascade, broadcast, and multiple-access networks, in Subsections IV-A, IV-B, and IV-C, respectively. In Section V, we discuss the implications of our results on quantum nonlocal games. The analysis for the three networks with classical links is given in Sections VI, VII, VIII, and for the three networks with quantum links in Sections IX, X, XI. Section XII is dedicated to summary and discussion.

II. Notation

We use standard notation in quantum information theory, as in [38], $X, Y, Z, \ldots$ are discrete random variables on finite alphabets $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, ..., respectively, The distribution of X is specified by a probability mass function (pmf) $p_X(x)$ on $\mathcal{X}$. The set of all pmfs over $\mathcal{X}$ is denoted by $\mathcal{P}(\mathcal{X})$. We use $x^n = (x_i)_{i\in[n]}$ denotes for a sequence in of letters from $\mathcal{X}^n$. A quantum state of a quantum system A is described by a density operator, ρ_A, on the Hilbert space $\mathcal{H}_A$. Denote the set of all such operators by $\Delta(\mathcal{H}_A)$. A c-q channel is a map $\mathcal{N}_{X\to B} : \mathcal{X} \to \Delta(\mathcal{H}_B)$. A measurement is specified by a collection of operators $\{D_j\}$ that forms a POVM. positive operator-valued measure (POVM). , i.e., $D_j \geq 0$ and $\sum_j D_j = \mathbb{1}$, where $\mathbb{1}$ is the identity operator. Given a bipartite state ρ_{AB}, on $\mathcal{H}_A \otimes \mathcal{H}_B$, the quantum mutual information is defined as $I(A; B)_\rho = H(\rho_A) + H(\rho_B) - H(\rho_{AB})$, where $H(\rho) \equiv -\mathrm{Tr}[\rho \log(\rho)]$, is the von Neumann entropy, the conditional quantum entropy as $H(A|B)_\rho = H(\rho_{AB}) - H(\rho_B)$, and $I(A; B|C)_\rho = H(A|C)_\rho + H(B|C)_\rho - H(A, B|C)_\rho$, is defined accordingly.

III. Classical links — Model Definitions and Results

We begin with networks with classical links. We consider three coordination settings with classical communication links as described below.

A. Two-Node Network

Consider the two-node network in Figure 9. Here, we use simpler notation, $R_1 \equiv R_{1,2}$, for convenience. Alice and Bob wish to simulate a c-q state $\omega_{XB}^{\otimes n}$, using the following scheme. Node 1 (Alice) receives a classical source sequence x^n, drawn by Nature according to a given PMF p_X. The source sequence is encoded into an index m_1 at a rate R_1. Node 2 (Bob) is quantum. Both nodes have access to a CR element m_0 at a given rate R_0, i.e., m_0 is uniformly distributed over $\left[2^{nR_0}\right]$, and it is independent of X^n.

Formally, a $\left(2^{nR_0}, 2^{nR_1}, n\right)$ coordination code for the simulation of a c-q state ω_{XB} consists of a classical encoding channel, $F : \mathcal{X}^n \times [2^{nR_0}] \to [2^{nR_1}]$, and a c-q decoding channel $\mathcal{D}_{M_0 M_1 \to B^n}$. The protocol works as follows. A classical sequence $x^n \sim p_X^n$ is generated by Nature. Given the sequence x^n and the CR element m_0, Alice selects a random index,

$$m_1 \sim F(\cdot|x^n, m_0) \tag{16}$$

and sends it through a noiseless link. As Bob receives the message m_1 and the CR element m_0, he prepares the state

$$\rho_{B^n}^{(m_0,m_1)} = \mathcal{D}_{M_0 M_1 \to B^n}(m_0, m_1)\,. \tag{17}$$

Hence, the resulting joint state is

$$\widehat{\rho}_{X^n B^n} = \frac{1}{2^{nR_0}} \sum_{m_0 \in [2^{nR_0}]} \sum_{x^n \in \mathcal{X}^n} \left(p_X^n(x^n) \, |x^n\rangle\!\langle x^n|_{X^n} \otimes \sum_{m_1 \in [2^{nR_1}]} F(m_1|x^n, m_0) \rho_{B^n}^{(m_0,m_1)} \right). \tag{18}$$

Definition 1. A coordination rate pair (R_0, R_1) is achievable for the simulation of ω_{XB}, if for every $\varepsilon, \delta > 0$ and sufficiently large n, there exists a $\left(2^{n(R_0+\delta)}, 2^{n(R_1+\delta)}, n\right)$ code that achieves

$$\left\| \widehat{\rho}_{X^n B^n} - \omega_{XB}^{\otimes n} \right\|_1 \leq \varepsilon\,. \tag{19}$$

The coordination capacity region of the two-node network, $\mathcal{R}_{\text{2-node}}(\omega)$, with respect to the c-q state ω_{XB}, is the closure of the set of all achievable rate pairs.

The coordination capacity, $\mathsf{C}_{\text{2-node}}^{(0)}(\omega)$, without CR, is the supremum of rates R_1 such that $(0, R_1) \in \mathcal{R}_{\text{2-node}}(\omega)$. The CR-assisted coordination capacity, $\mathsf{C}_{\text{2-node}}^{(\infty)}(\omega)$, i.e., with unlimited CR, is the supremum of rates R_1 such that $(R_0, R_1) \in \mathcal{R}_{\text{2-node}}(\omega)$ for some $R_1 \geq 0$.

The optimal coordination rates for the two-node network are established below.

Consider a given c-q state ω_{XB} that we wish to simulate. We now state our main result. Define the following set of c-q states. Let $\mathscr{S}_{\text{2-node}}(\omega)$ be the set of all c-q states

$$\sigma_{XUB} = \sum_{\substack{(x,u)\in \\ \mathcal{X}\times\mathcal{U}}} p_{X,U}(x,u) \, |x\rangle\!\langle x|_X \otimes |u\rangle\!\langle u|_U \otimes \theta_B^u \tag{20a}$$

such that

$$\sigma_{XB} = \omega_{XB} \tag{20b}$$

for $|\mathcal{U}| \leq |\mathcal{X}|^2 [\dim(\mathcal{H}_B)]^2 + 1$. Notice that given a classical value $U = u$, there is no correlation between X and B.

Theorem 1. The coordination capacity region for the two-node network described in Figure 9 is given by the set

$$\mathcal{R}_{\text{2-node}}(\omega) = \bigcup_{\mathscr{S}_{\text{2-node}}(\omega)} \left\{ \begin{array}{rl} (R_0, R_1) \in \mathbb{R}^2 : \quad R_1 & \geq I(X;U)_\sigma\,, \\ R_0 + R_1 & \geq I(XB;U)_\sigma \end{array} \right\}. \tag{21}$$

The proof for Theorem 1 is given in Section VI. The following corollaries immediately follow.

Corollary 2 (Quantum Common Information [8]). The coordination capacity without CR is

$$\mathsf{R}_{\text{2-node}}^{(0)}(\omega) = \min_{\sigma_{XUB} \in \mathscr{S}_{\text{2-node}}(\omega)} I(XB;U)_\sigma\,. \tag{22}$$

Corollary 3. The CR-assisted coordination capacity, i.e., with unlimited common randomness, is given by

$$\mathsf{R}_{\text{2-node}}^{(\infty)}(\omega) \triangleq \min_{\sigma_{XUB} \in \mathscr{S}_{\text{2-node}}(\omega)} I(X;U)_\sigma \tag{23}$$

We note that in order to achieve the CR-assisted capacity, a CR rate of $R_0 = I(U;B|X)_\sigma$ is sufficient. If $B \equiv Y$ is classical, then we may substitute $U = Y$, which yields the capacity $\mathsf{R}_{\text{2-node}}^{(\infty)}(\omega) = I(X;Y)$, and it can be achieved with CR at rate $R_0 = H(Y|X)$ [26].

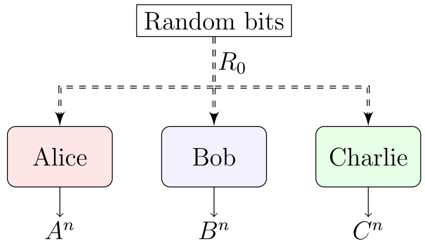


Fig. 10: No-communication network

B. No-Communication Network

Consider a network that consists of three users: Alice, Bob and Charlie, holding quantum systems A, B, and C, respectively. The users cannot communicate, but they share a CR element m_0 at a rate R_0, as illustrated in Figure 10. Given m_0, each user prepares a quantum state separately.

A $\left(2^{nR_0}, n\right)$ coordination code for the no-communication network consists of a CR set $[2^{nR_0}]$, and three c-q encoding channels, $\mathcal{T}^{(1)}_{M_0\to A^n}$, $\mathcal{T}^{(2)}_{M_0\to B^n}$, and $\mathcal{T}^{(3)}_{M_0\to C^n}$. As Alice, Bob, and Charlie receive a realization j of the CR element, each uses their encoding map to prepare their respective state. prepares a quantum state, $\rho^j_{A^n} = \mathcal{T}^{(1)}_{M_0\to A^n}(m_0)$, $\rho^j_{B^n} = \mathcal{T}^{(2)}_{M_0\to A^n}(m_0)$, and $\rho^j_{C^n} = \mathcal{T}^{(3)}_{M_0\to C^n}(m_0)$, respectively. Hence,

$$\widehat{\rho}_{A^nB^nC^n} = \frac{1}{2^{nR_0}} \sum_{m_0\in[2^{nR_0}]} \mathcal{T}^{(1)}(m_0)\otimes\mathcal{T}^{(2)}(m_0)\otimes\mathcal{T}^{(3)}(m_0)\,. \tag{24}$$

Definition 2. A CR rate R_0 is achievable for the simulation of ω_{ABC}, if for every $\varepsilon, \delta > 0$ and sufficiently large n, there exists a $\left(2^{n(R_0+\delta)}, n\right)$ coordination code that achieves

$$\left\|\widehat{\rho}_{A^nB^nC^n} - \omega^{\otimes n}_{ABC}\right\|_1 \le \varepsilon\,. \tag{25}$$

The coordination capacity $\mathsf{C}_{\text{NC}}(\omega)$, for the no-communication network, is the infimum of achievable rates R_0. If there are no achievable rates, we set $\mathsf{C}_{\text{NC}}(\omega) = +\infty$.

The optimal coordination rates for the no-communication network are established below. Consider a given quantum state ω_{ABC} that we wish to simulate. We now state our main result. Define the following set of c-q states. Let $\mathscr{S}_{\text{NC}}(\omega)$ be the set of all c-q states

$$\sigma_{UABC} = \sum_{u\in\mathcal{U}} p_U(u)\ |u\rangle\langle u|_U \otimes \theta^u_A\otimes\theta^u_B\otimes\theta^u_C \tag{26a}$$

such that

$$\sigma_{ABC} = \omega_{ABC} \tag{26b}$$

Given $U = u$, there is no correlation between A, B and C.

Theorem 4. The coordination capacity for the no-communication network described in Figure 10 is

$$\mathsf{C}_{\text{NC}}(\omega) = \inf_{\sigma_{UABC}\in\mathscr{S}_{\text{NC}}(\omega)} I(U;ABC)_\sigma \tag{27}$$

with the convention that an infimum over an empty set is $+\infty$.

The proof for Theorem 4 is given in Section VII.

Remark 1. Since the CR is classical, it cannot be used in order to create entanglement. Therefore, as Alice, Bob, and Charlie do not cooperate with one another, it is impossible to simulate entanglement. That is, we can only simulate separable states.

Remark 2. For a product state $\omega_{ABC} = \omega_A\otimes\omega_B\otimes\omega_C$, we may take U to be null, hence $\mathsf{C}_{\text{NC}}(\omega) = 0$. That is, in the case of a product state, the simulation does not require CR between the users. On the other hand, if ω_{AB} is entangled, for instance, then there is no U that can satisfy (26), thus $\mathsf{C}_{\text{NC}}(\omega) = +\infty$.

For a classically correlated state $\omega_{ABC} = \frac{1}{2}\left(|000\rangle\langle 000| + |111\rangle\langle 111|\right)$, we have $\mathsf{C}_{\text{NC}}(\omega) = 1$, since one bit of CR is required in order to simulate such correlation. This rate is achieved with $U \sim \text{Bernoulli}\left(\frac{1}{2}\right)$.

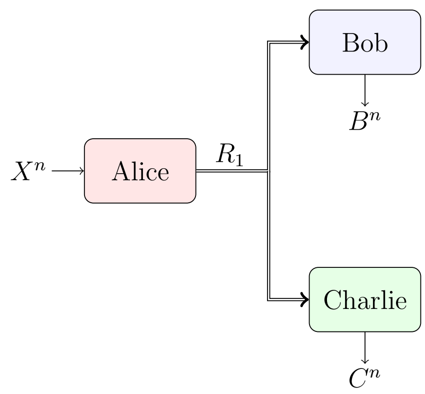


Fig. 11: Broadcast Network. The CR element is omitted for simplicity.

C. Broadcast Network

Consider the broadcast network in Figure 11. A sender, Alice, and two receivers, Bob and Charlie, wish to simulate a c-q-q state ω_{XBC}, using the following scheme. Alice receives a classical source sequence $x^n \in \mathcal{X}^n$ drawn by Nature, i.i.d. according to a given PMF p_X. Alice encodes the source sequence into an index m_1 at a rate R_1. The other two nodes, of Bob and Charlie, are quantum. The three nodes have access to a CR element m_0 at a rate R_0. Similarly, a $\left(2^{nR_0}, 2^{nR_1}, n\right)$ coordination code consists of a classical encoding channel, $F : \mathcal{X}^n \times [2^{nR_0}] \to [2^{nR_1}]$, and two c-q decoding channels, $\mathcal{D}^{(\ell)}_{M_0 M_1 \to B^n_\ell}$, for $\ell \in \{1, 2\}$. Given x^n and the CR element m_0, Alice generates $m_1 \sim F(\cdot|x^n, m_0)$, and sends it to both Bob and Charlie, who then apply their decoding map.

The coordination capacity region of the broadcast network, $\mathcal{R}_{\text{BC}}(\omega)$, with respect to the c-q state ω_{XBC}, is defined in a similar manner as in Definition 1.

The optimal coordination rates for the broadcast network with classical links are established below.

Consider a given c-q-q state ω_{XBC} that we wish to simulate. Define the following set of c-q-q states. Let $\mathscr{S}_{\text{2-BC}}(\omega)$ be the set of all c-q states

$$\sigma_{XUBC} = \sum_{\substack{(x,u)\in\\ \mathcal{X}\times\mathcal{U}}} p_{XU}(x,u) \, |x\rangle\langle x|_X \otimes |u\rangle\langle u|_U \otimes \theta_B^u \otimes \eta_C^u \tag{28}$$

such that

$$\sigma_{XBC} = \omega_{XBC} \, . \tag{29}$$

Note that given X, B, and C are uncorrelated given $U = u$.

Theorem 5. The coordination capacity region of the broadcast network in Figure 11 is the set

$$\mathcal{R}_{\text{BC}}(\omega) = \bigcup_{\mathscr{S}_{\text{BC}}(\omega)} \left\{ \begin{array}{rl} (R_0, R_1) \in \mathbb{R}^2 : & \begin{array}{rl} R_1 & \geq I(X;U)_\sigma \, , \\ R_0 + R_1 & \geq I(XBC;U)_\sigma \end{array} \end{array} \right\} . \tag{30}$$

The proof for Theorem 5 is given in Section VIII. The following corollaries immediately follow.

Remark 3. Since Alice's encoding is classical, she cannot distribute entanglement. Therefore, as Bob and Charlie do not cooperate with one another, it is impossible to simulate entanglement between Bob and Charlie. That is, we can only simulate states such that ω_{BC} is separable, as in the no-communication model (see Remark 1).

IV. Quantum links - Model Definitions and Results

We consider three coordination settings with quantum communication links, as described below. We then discuss the implications of the results obtained for the broadcast network shown in Subsection IV-B on nonlocal games.

A. Cascade network

Consider the cascade network with rate-limited entanglement, as depicted in Figure 12. In the Introduction section, we used the notation $Q_{i,j}$ for the communication rate from Node i to Node j. Here, we simplify the notation, and write $Q_1 \equiv Q_{1,2}$ and $Q_2 \equiv Q_{2,3}$, for convenience.

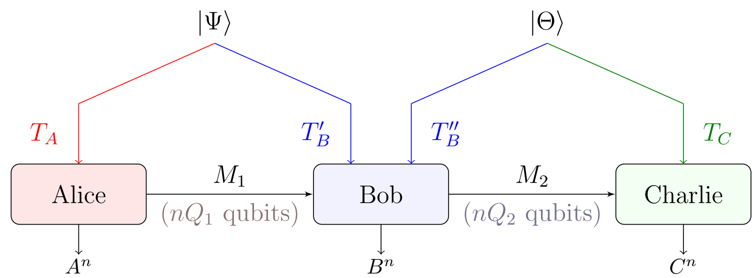


Fig. 12: Cascade network with rate-limited entanglement.

Alice, Bob, and Charlie would like to simulate a joint state $\omega_{ABC}^{\otimes n}$, where $\omega_{ABC} \in \Delta(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$. Before communication begins, each party shares bipartite entanglement with their nearest neighbor. The bipartite state $|\Psi_{T_A T'_B}\rangle$ indicates the entanglement resource shared between Alice and Bob, while $|\Theta_{T''_B T_C}\rangle$ is shared between Bob and Charlie. The coordination protocol begins with Alice preparing the state of her output system A^n, as well as a "quantum description" M_1. She sends M_1 to Bob. As Bob receives M_1, he encodes the output B^n, along with his own quantum description, M_2. Next, Bob sends M_2 to Charlie. Upon receiving M_2, Charlie prepares the output state for C^n.

The transmissions M_1 and M_2 are limited to the quantum communication rates Q_1 and Q_2, while the pre-shared resources between Alice and Bob and between Bob and Charlie are limited to the entanglement rates E_1 and E_2, respectively.

Definition 3. A $(2^{nQ_1}, 2^{nQ_2}, 2^{nE_1}, 2^{nE_2}, n)$ coordination code for the cascade network in Figure 12 consists of:

- Two bipartite states $|\Psi_{T_A T'_B}\rangle$ and $|\Theta_{T''_B T_C}\rangle$ on Hilbert spaces of dimension 2^{nE_1} and 2^{nE_2}, respectively, i.e.,

$$\dim(\mathcal{H}_{T_A}) = \dim(\mathcal{H}_{T'_B}) = 2^{nE_1}\,, \tag{31}$$
$$\dim(\mathcal{H}_{T''_B}) = \dim(\mathcal{H}_{T_C}) = 2^{nE_2}\,, \tag{32}$$

- two Hilbert spaces, $\mathcal{H}_{M_1}$ and $\mathcal{H}_{M_2}$, of dimension

$$\dim(\mathcal{H}_{M_j}) = 2^{nQ_j} \quad \text{for } j \in \{1,2\}\,, \tag{33}$$

  and
- three encoding maps,

$$\mathcal{E}_{T_A \to A^n M_1} : \Delta(\mathcal{H}_{T_A}) \to \Delta(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_{M_1})\,, \tag{34}$$
$$\mathcal{F}_{M_1 T'_B T''_B \to B^n M_2} : \Delta(\mathcal{H}_{M_1} \otimes \mathcal{H}_{T'_B T''_B}) \to \Delta(\mathcal{H}_B^{\otimes n} \otimes \mathcal{H}_{M_2})\,, \tag{35}$$

  and

$$\mathcal{D}_{M_2 T_C \to C^n} : \Delta(\mathcal{H}_{M_2} \otimes \mathcal{H}_{T_C}) \to \Delta(\mathcal{H}_C^{\otimes n})\,, \tag{36}$$

  corresponding to Alice, Bob, and Charlie, respectively.

The coordination protocol has limited communication rates Q_j and entanglement rates E_j, for $j \in \{1,2\}$. That is, before the protocol begins, Alice and Bob are provided with nE_1 qubit pairs, while Bob and Charlie share nE_2 pairs. During the protocol, Alice transmits nQ_1 qubits to Bob, and then Bob transmits nQ_2 qubits to Charlie. See Figure 12. A detailed description of the protocol is given below.

The coordination protocol works as follows. Alice applies the encoding map $\mathcal{E}_{T_A \to A^n M_1}$ on her share T_A of the entanglement resources. This results in the output state

$$\rho^{(1)}_{A^n M_1 T'_B} = (\mathcal{E}_{T_A \to A^n M_1} \otimes \mathrm{id}_{T'_B})(\Psi_{T_A T'_B})\,. \tag{37}$$

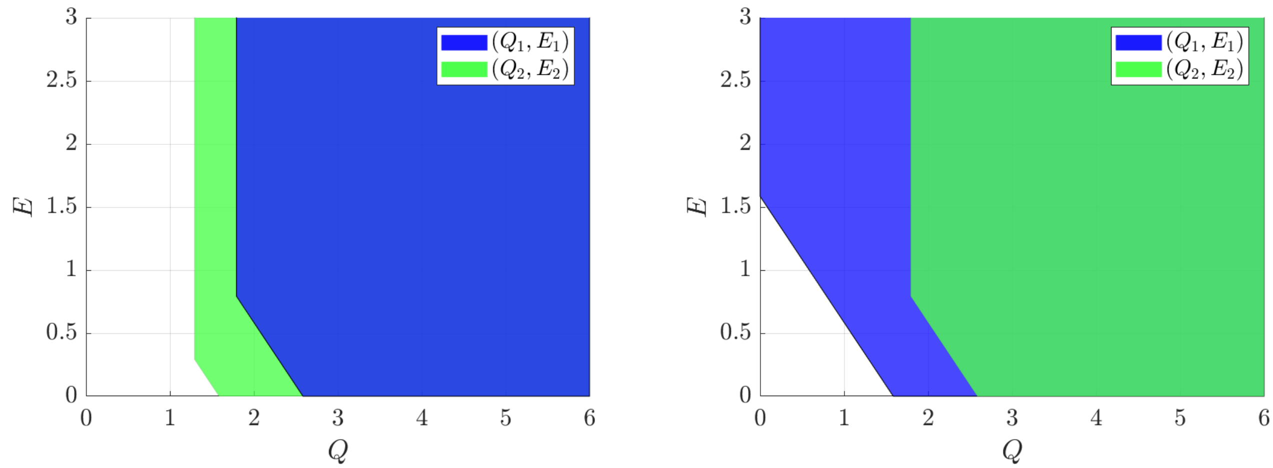
She sends M_1 to Bob. Having received M_1, Bob uses it along with his share $T'_B T''_B$ of the entanglement resources to encode the systems B^n and M_2. To this end, he uses the map $\mathcal{F}_{M_1 T'_B T''_B \to B^n M_2}$, hence

$$\rho^{(2)}_{A^n B^n M_2 T_C} = (\mathrm{id}_{A^n} \otimes \mathcal{F}_{M_1 T'_B T''_B \to B^n M_2} \otimes \mathrm{id}_{T_C})(\rho^{(1)}_{A^n M_1 T'_B} \otimes \Theta_{T''_B T_C})\,. \tag{38}$$

Bob sends M_2 to Charlie, who applies the encoding channel $\mathcal{D}_{M_2 T_C \to C^n}$. This results in the final joint state,

$$\widehat{\rho}_{A^n B^n C^n} = (\mathrm{id}_{A^n B^n} \otimes \mathcal{D}_{M_2 T_C \to C^n})\left(\rho^{(2)}_{A^n B^n M_2 T_C}\right)\,. \tag{39}$$

The objective is that the final state $\widehat{\rho}_{A^n B^n C^n}$ is arbitrarily close to the desired state $\omega_{ABC}^{\otimes n}$.



(a) Example 1: Mixture (b) Example 2: Entanglement

Fig. 13: Coordination capacity region in two examples.

Definition 4. A rate tuple (Q_1, Q_2, E_1, E_2) is achievable, if for every $\varepsilon, \delta > 0$ and sufficiently large n, there exists a $(2^{n(Q_1+\delta)}, 2^{n(Q_2+\delta)}, 2^{n(E_1+\delta)}, 2^{n(E_2+\delta)}, n)$ coordination code satisfying

$$\left\| \widehat{\rho}_{A^n B^n C^n} - \omega_{ABC}^{\otimes n} \right\|_1 \leq \varepsilon. \tag{40}$$

The coordination capacity region with respect to the state ω_{ABC} is defined as the closure of the set of all achievable rate tuples. We denote the coordination capacity region of the quantum cascade network, with quantum links and rate-limited entanglement, by $\mathcal{Q}_{\text{Cascade}}(\omega)$.

Remark 4. Coordination in the cascade network can also be represented as as a resource inequality [39]

$$Q_1[q \to q]_{A \to B} + E_1[qq]_{AB} + Q_2[q \to q]_{B \to C} + E_2[qq]_{BC} \geq \langle \omega_{ABC} \rangle \tag{41}$$

where the resource units $[q \to q]$, $[qq]$, and $\langle \omega_{ABC} \rangle$ represent a single use of a noiseless qubit channel, an EPR pair, and the desired state ω_{ABC}, respectively.

The optimal coordination rates for the cascade network are established below.

Theorem 6. Let $|\omega_{RABC}\rangle$ be a purification of ω_{ABC}. The coordination capacity region for the cascade network described in Figure 12 is given by the set

$$\mathcal{Q}_{\text{Cascade}}(\omega) = \left\{ \begin{array}{rrl} (Q_1, E_1, Q_2, E_2): & Q_1 & \geq \frac{1}{2} I(BC; R)_\omega , \\ & Q_1 + E_1 & \geq H(BC)_\omega , \\ & Q_2 & \geq \frac{1}{2} I(C; RA)_\omega , \\ & Q_2 + E_2 & \geq H(C)_\omega \end{array} \right\} . \tag{42}$$

The proof for Theorem 6 is provided in Section IX.

Corollary 7. For a pure state $|\omega_{ABC}\rangle$, The coordination capacity region for the cascade network is given by the set

$$\mathcal{Q}_{\text{Cascade}}(\omega) = \left\{ \begin{array}{rrl} (Q_1, E_1, Q_2, E_2): & Q_1 + E_1 & \geq H(BC)_\omega , \\ & Q_2 & \geq \frac{1}{2} I(C; A)_\omega , \\ & Q_2 + E_2 & \geq H(C)_\omega \end{array} \right\} . \tag{43}$$

The examples below demonstrate that coordination of entanglement and coordination of separable correlations behave differently.

Example 1 (Mixture). Let $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_C$ be Hilbert spaces of dimension 3, i.e., qutrits. Consider the simulation of a mixed state,

$$\omega_{ABC} = \frac{1}{6} \left(|012\rangle\langle 012| + |021\rangle\langle 021| + |102\rangle\langle 102| + |120\rangle\langle 120| + |201\rangle\langle 201| + |210\rangle\langle 210| \right) \tag{44}$$

The example is analogous to [40, Example 3]. The state above is thus purified by

$$|\omega_{ABCR}\rangle = \frac{1}{6} \big(|012\rangle \otimes |0\rangle + |021\rangle \otimes |1\rangle + |102\rangle \otimes |2\rangle + |120\rangle \otimes |3\rangle + |201\rangle \otimes |4\rangle + |210\rangle \otimes |5\rangle \big) \tag{45}$$

where $\{|i\rangle\}_{i=0,\dots,5.}$ forms an orthonormal basis for the reference system R. In this case, the coordination capacity region is given by

$$\mathcal{Q}_{\text{Cascade}}(\omega) = \left\{ \begin{array}{rl} (Q_1, E_1, Q_2, E_2) : \qquad Q_1 & \geq 1.7925, \\ Q_1 + E_1 & \geq 2.5850, \\ Q_2 & \geq 1.2925, \\ Q_2 + E_2 & \geq 1.5850. \end{array} \right\} . \tag{46}$$

The coordination capacity region $\mathcal{Q}_{\text{Cascade}}(\omega)$ is illustrated in Figure 13 (a), where the blue region shows the tradeoff between Alice's rates, Q_1 and E_1, and the green region is associated with Bob's rates, Q_2 and E_2.

Suppose that $E_1 = E_2$. As can be seen in the figure, Alice is required to send qubits to Bob at a higher rate than Bob to Charlie. This is intuitive since Alice encodes information for both Bob and Charlie, whereas Bob is only encoding Charlie's information.

Example 2 (Entanglement). Consider the simulation of a pure tripartite entangled state,

$$|\psi_{ABC}\rangle = \frac{1}{\sqrt{6}} (|012\rangle + |021\rangle + |102\rangle + |120\rangle + |201\rangle + |210\rangle) \tag{47}$$

According to Corollary 7, $|\psi_{ABC}\rangle^{\otimes n}$ can be simulated if and only if the rate tuple (Q_1, E_1, Q_2, E_2) belongs to the following set,

$$\mathcal{Q}_{\text{Cascade}}(\psi) = \left\{ \begin{array}{rl} (Q_1, E_1, Q_2, E_2) : \quad Q_1 + E_1 & \geq 1.5850\,, \\ Q_2 & \geq 0.7925\,, \\ Q_2 + E_2 & \geq 1.5850\,. \end{array} \right\} .$$

The coordination capacity region $\mathcal{Q}_{\text{Cascade}}(\psi)$ is illustrated in Figure 13 (b). As before, the blue region shows the tradeoff between Alice's rates, Q_1 and E_1, and the green region is associated with Bob's rates, Q_2 and E_2.

Suppose that $E_1 = E_2$. Here, as opposed to Example 1, Alice is required to send qubits to Bob at a rate that is *lower* than Bob to Charlie. This occurs because of the "knowing less than nothing" phenomenon [41]. That is, in the presence of entanglement, a subsystem can have a larger entropy compared to the joint system. The behavior in each example is completely different.

B. *Broadcast network*

Consider the broadcast network in Figure 14. This network, can be useful in analyzing refereed games and the required resources for achieving certain performances as described in section V. As before, we simplify the notation $Q_{i,j}$ from the Introduction section, and write $Q_1 \equiv Q_{1,2}$ and $Q_2 \equiv Q_{1,3}$, for convenience. Consider a classical-quantum state,

$$\omega_{XYABC} = \sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} p_{XY}(x,y) \, |x,y\rangle\langle x,y|_{X,Y} \otimes \left|\sigma_{ABC}^{(x,y)}\right\rangle\!\left\langle\sigma_{ABC}^{(x,y)}\right| \tag{48}$$

corresponding to a given ensemble of states $\left\{p_{XY}, \left|\sigma_{ABC}^{(x,y)}\right\rangle\right\}$ in $\Delta(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$.

Alice, Bob, and Charlie would like to simulate ω_{XYABC}. Before communication takes place, the classical sequences X^n and Y^n are drawn from a common source $p_{XY}^{\otimes n}$. The sequence X^n is given to Bob, while Y^n is given to Charlie (see Figure 14).

Initially, Alice encodes her output A^n, along with two quantum descriptions, M_1 and M_2. She then transmits M_1 and M_2, to Bob and Charlie, respectively, at limited qubit transmission rates, Q_1 and Q_2. As Bob receives the quantum description M_1, he uses it together with the classical sequence X^n to encode the output B^n. Similarly, Charlie receives M_2 and Y^n, and encodes his output C^n.

Definition 5. A $(2^{nQ_1}, 2^{nQ_2}, n)$ coordination code for the broadcast network with side information described in Figure 14, consists of two Hilbert spaces, $\mathcal{H}_{M_1}$ and $\mathcal{H}_{M_2}$, of dimensions

$$\dim(\mathcal{H}_{M_j}) = 2^{nQ_j} \text{ for } j \in \{1,2\} \ , \tag{49}$$

and three encoding maps,

$$\mathcal{E}_{A^n \to A^n M_1 M_2} : \Delta(\mathcal{H}_A^{\otimes n}) \to \Delta(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_{M_1} \otimes \mathcal{H}_{M_2}), \tag{50}$$
$$\mathcal{F}_{X^n M_1 \to B^n} : \mathcal{X}^n \otimes \Delta(\mathcal{H}_{M_1}) \to \Delta(\mathcal{H}_B^{\otimes n}), \tag{51}$$

and

$$\mathcal{D}_{Y^n M_2 \to C^n} : \mathcal{Y}^n \otimes \Delta(\mathcal{H}_{M_2}) \to \Delta(\mathcal{H}_C^{\otimes n}). \tag{52}$$

corresponding to Alice, Bob, and Charlie, respectively. In the course of the protocol, Alice transmits nQ_1 qubits to Bob and nQ_2 qubits to Charlie, as illustrated in Figure 14.

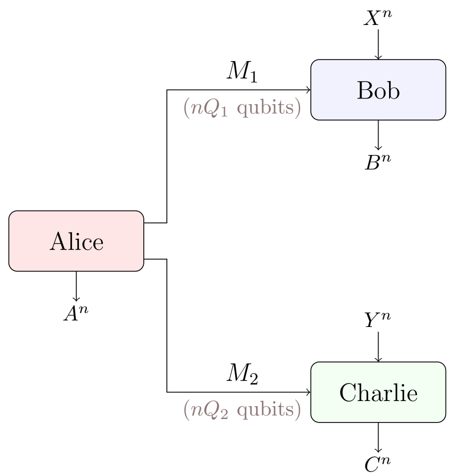


Fig. 14: Broadcast network.

Remark 5. In the quantum world, broadcasting a quantum state among multiple receivers is impossible by the no-cloning theorem. However, in the broadcast network in Figure 14, Alice sends two different "quantum messages" M_1 and M_2 to Bob and Charlie, respectively. Roughly speaking, Alice is broadcasting correlation. Since Alice prepares both quantum descriptions, M_1 and M_2, she can create correlation and generate tripartite entanglement between her, Bob, and Charlie.

The coordination protocol is described below. Alice applies her encoding map and prepares

$$\rho^{(1)}_{A^n M_1 M_2} = \mathcal{E}_{A^n \to A^n M_1 M_2}(\omega_A^{\otimes n})\,. \tag{53}$$

She sends M_1 and M_2 to Bob and Charlie, respectively. Once Bob receives M_1 and the classical assistance, X^n, he applies his encoding map $\mathcal{F}_{X^n M_1 \to B^n}$. Similarly, Charlie receives M_2 and Y^n, and applies $\mathcal{D}_{Y^n M_2 \to C^n}$. Their encoding operations result in the following extended state:

$$\widehat{\rho}_{X^n Y^n A^n B^n C^n} = \sum_{x^n \in \mathcal{X}^n} \sum_{y^n \in \mathcal{Y}^n} p_{XY}^{\otimes n}(x^n, y^n)\, |x^n, y^n\rangle\langle x^n, y^n|_{X^n Y^n} \otimes \\ (\mathrm{id}_{A^n} \otimes \mathcal{F}_{X^n M_1 \to B^n} \otimes \mathcal{D}_{Y^n M_2 \to C^n}) \left(|x^n, y^n\rangle\langle x^n, y^n|_{\bar{X}^n \bar{Y}^n} \otimes \rho^{(1)}_{A^n M_1 M_2} \right), \tag{54}$$

where $\bar{X}^n \bar{Y}^n$ are classical registers that store a copy of the (classical) sequences $X^n Y^n$, respectively. The goal is to encode such that the final state $\widehat{\rho}_{X^n Y^n A^n B^n C^n}$ is arbitrarily close to the desired state $\omega_{XYABC}^{\otimes n}$.

Definition 6. A rate pair (Q_1, Q_2) is achievable, if for every $\varepsilon, \delta > 0$ and sufficiently large n, there exists a $(2^{n(Q_1+\delta)}, 2^{n(Q_2+\delta)}, n)$ coordination code satisfying

$$\left\| \widehat{\rho}_{X^n Y^n A^n B^n C^n} - \omega_{XYABC}^{\otimes n} \right\|_1 \leq \varepsilon. \tag{55}$$

The coordination capacity region of the broadcast network, $\mathcal{Q}_{\mathrm{BC}}(\omega)$, with respect to the state ω_{XYABC}, is the closure of the set of all achievable rate pairs.

Remark 6. Notice that Alice has no access to X^n nor Y^n. Therefore, coordination can only be achieved for states ω_{XYABC} such that there is no correlation between A and XY, on their own. That is, the reduced state ω_{XYA} must have a product form,

$$\omega_{XYA} = \omega_{XY} \otimes \omega_A\,. \tag{56}$$

Since Alice does not share prior correlation with Bob and Charlie's resources X^n and Y^n, standard techniques, such as state redistribution [12] and quantum source coding with side information [36], are not suitable for our purposes. Instead, we introduce a quantum version of binning.

The optimal coordination rates for the broadcast network are established below.

Theorem 8. The coordination capacity region for the broadcast network described in Figure 14 is given by the set

$$\mathcal{Q}_{\mathrm{BC}}(\omega) = \left\{ \begin{array}{lll} (Q_1, Q_2) \in \mathbb{R}^2 : & Q_1 & \geq H(B|X)_\omega\,, \\ & Q_2 & \geq H(C|Y)_\omega \end{array} \right\}. \tag{57}$$

The proof for Theorem 8 is provided in Section X. The implications of this result on quantum nonlocal games are discussed in Section V.

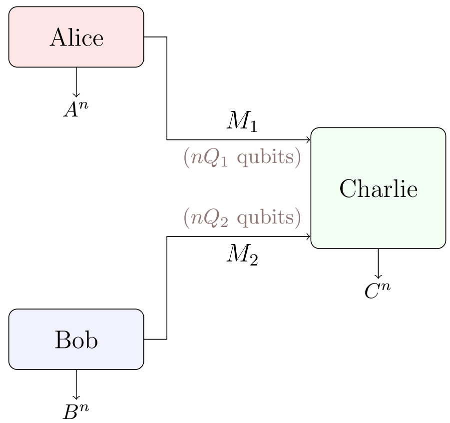


Fig. 15: Multiple access network.

C. Multiple access network

Consider the multiple-access network in Figure 15. Alice, Bob, and Charlie would like to simulate a pure state $|\omega_{ABC}\rangle^{\otimes n}$, where $|\omega_{ABC}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. We simplify the notation and write $Q_1 \equiv Q_{1,3}$ and $Q_2 \equiv Q_{2,3}$. At first, Alice prepares the state of the quantum systems A^n and M_1, and Bob prepares the states of the quantum systems B^n and M_2. Alice and Bob send M_1 and M_2 to Charlie. Charlie then uses M_1 and M_2 to encode the system C^n. As in the previous settings, M_1 and M_2 are referred to as quantum descriptions, which are limited to the qubit transmission rates, Q_1 and Q_2, respectively.

Definition 7. A $(2^{\ell_1}, 2^{\ell_2}, n)$ coordination code for the multiple-access network described in Figure 15, consists of two Hilbert spaces, $\mathcal{H}_{M_1}$ and $\mathcal{H}_{M_2}$, of dimensions

$$\dim(\mathcal{H}_{M_j}) = 2^{\ell_j} \text{ for } j \in \{1,2\} \ , \tag{58}$$

and three encoding maps,

$$\mathcal{E}_{A^n \to A^n M_1} : \Delta(\mathcal{H}_A^{\otimes n}) \to \Delta(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_{M_1}) \,, \tag{59}$$
$$\mathcal{F}_{B^n \to B^n M_2} : \Delta(\mathcal{H}_B^{\otimes n}) \to \Delta(\mathcal{H}_B^{\otimes n} \otimes \mathcal{H}_{M_2}) \tag{60}$$

and

$$\mathcal{D}_{M_1 M_2 \to C^n} : \Delta(\mathcal{H}_{M_1} \otimes \mathcal{H}_{M_2}) \to \Delta(\mathcal{H}_C^{\otimes n}) \,, \tag{61}$$

corresponding Alice, Bob, and Charlie, respectively.

In the multiple-access network, Alice sends nQ_1 qubits to Charlie, while Bob sends nQ_2 qubits to Charlie. Specifically, Alice and Bob apply the encoding maps, preparing $\rho^{(1)}_{A^n M_1} \otimes \rho^{(2)}_{B^n M_2}$, where

$$\rho^{(1)}_{A^n M_1} = \mathcal{E}_{A^n \to A^n M_1}(\omega_A^{\otimes n}) \,,\ \rho^{(2)}_{B^n M_2} = \mathcal{F}_{B^n \to B^n M_2}(\omega_B^{\otimes n}) \,. \tag{62}$$

As Charlie receives M_1 and M_2, he applies his encoding map, which yields the final state,

$$\widehat{\rho}_{A^n B^n C^n} = (\mathrm{id}_{A^n B^n} \otimes \mathcal{D}_{M_1 M_2 \to C^n}) \, (\rho^{(1)}_{A^n M_1} \otimes \rho^{(2)}_{B^n M_2}) \,. \tag{63}$$

The ultimate goal of the coordination protocol is that the final state of $\widehat{\rho}_{A^n B^n C^n}$, is arbitrarily close to the desired state $\omega_{ABC}^{\otimes n}$.

Remark 7. Notice that since Charlie only acts on M_1 and M_2 which are encoded separately without coordination, we have $\widehat{\rho}_{A^n B^n} = \rho^{(1)}_{A^n} \otimes \rho^{(2)}_{B^n}$. Therefore, it is only possible to simulate states ω_{ABC} such that $\omega_{AB} = \omega_A \otimes \omega_B$. Since all purifications are isometrically equivalent [38, Theorem 5.1.1] there exists an isometry $V_{C \to C_1 C_2}$ such that

$$(\mathbb{1} \otimes V_{C \to C_1 C_2}) \, |\omega_{ABC}\rangle = |\phi_{AC_1}\rangle \otimes |\chi_{BC_2}\rangle \tag{64}$$

where $|\phi_{AC_1}\rangle$ and $|\chi_{BC_2}\rangle$ are purifications of ω_A and ω_B, respectively. If ω_{ABC} cannot be decomposed as in (64), then coordination is impossible in the multiple-access network.

Definition 8. A rate pair (Q_1, Q_2) is achievable, if for every $\varepsilon, \delta > 0$ and a sufficiently large n, there exists a $(2^{n(Q_1+\delta)}, 2^{n(Q_2+\delta)}, n)$ coordination code satisfying

$$\left\| \widehat{\rho}_{A^n B^n C^n} - \omega_{ABC}^{\otimes n} \right\|_1 \le \varepsilon \,. \tag{65}$$

The coordination capacity region of the multiple access network, $\mathcal{Q}_{\text{MAC}}(\omega)$, with respect to the state ω_{ABC}, is the closure of the set of all achievable rate pairs.

Remark 8. The resource inequality for coordination in the multiple-access network is

$$Q_1[q \to q]_{A\to C} + Q_2[q \to q]_{B\to C} \geq \langle \omega_{ABC} \rangle \tag{66}$$

(see resource definitions in Remark 4).

The optimal coordination rates for the multiple-access network are provided below.

Theorem 9. Let $|\omega_{ABC}\rangle$ be a pure state as in (64). The coordination capacity region for the multiple access network described in Figure 15 is given by the set

$$\mathcal{Q}_{\text{MAC}}(\omega) = \left\{ \begin{array}{lll} (Q_1, Q_2) \in \mathbb{R}^2 : & Q_1 & \geq H(A)_\omega \,, \\ & Q_2 & \geq H(B)_\omega \end{array} \right\} . \tag{67}$$

The proof for Theorem 9 is provided in Section XI.

V. Nonlocal Games

The broadcast network model presented in Figure 14 represents refereed games that have a quantum advantage. Such games are often referred to as nonlocal games [42]. The quantum advantage can be attributed to the quantum coordination between the users, before the beginning of the game.

First, we discuss the single-shot cooperative game, and then move on to sequential games with an asymptotic payoff. Consider Figure 14. Here, we assume that B and C are classical, while A is void. Here, Alice is a coordinator that generates correlation between the players, Bob and Charlie. In the sequential game, we denote the number of rounds by n.

Single shot game: The game involves a single round, hence $n = 1$. A referee provides two queries X and Y, drawn at random, one for Bob and the other for Charlie, respectively. The players, Bob and Charlie, provide responses, B and C, respectively. The players win the game (together) if the tuple (X, Y, B, C) satisfies a particular condition, $\mathscr{W}$. A well known example is the CHSH game [43], where $X, Y, B, C \in \{0, 1\}$, and the winning condition is

$$X \wedge Y = B \oplus C \,. \tag{68}$$

Using classical correlations, the game can be won with probability of at most 0.75.

If the players, Bob and Charlie, share a bipartite state $\rho_{M_1 M_2}$, then they can generate a quantum correlation,

$$P_{BC|XY}(b, c|xy) = \text{Tr}\left[\left(F_b^{(x)} \otimes D_c^{(y)}\right) \rho_{M_1 M_2}\right] \tag{69}$$

by performing local measurements $\{F_b^{(x)}\}$ and $\{D_c^{(y)}\}$, respectively. Such correlations can improve the players' performance. In particular, in the CHSH game, if the coordinator, Alice, provides the players with an EPR pair,

$$|\Phi_{M_1 M_2}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{70}$$

then their chance of winning improves to $\cos^2\left(\frac{\pi}{8}\right) \approx 0.8535$. As Alice sends one qubit to each player, $(Q_1, Q_2) = (1, 1)$ is optimal.

In pseudo-telepathy games, quantum strategies guarantee winning with probability 1. One example is the magic square game [44], where (X, Y) are the coordinates of a cell in the square, and the players win the game if they can provide 3 bits each that satisfy a parity condition. In this case, the game can be won with $Q_1 = Q_2 = 2$ qubits for each player. Slofstra and Vidick [45] presented a game where coordination of a correlation that could win with probability $(1 - e^{-T})$ requires $Q_j \propto T$ qubits for each user.

Sequential game: In the sequential version, the players repeat the game n times, and they can thus use a coordination code in order to play the game. In particular, Alice generates the entire correlation between the players a priori, before the sequential game begins. As the figure of merit, one may either consider the average chance of winning, or say, the minimal probability of winning. Let $\mathscr{S}(\gamma)$ denote the set of correlations $P_{BC|XY}$ that win the game with probability γ. Based on our results, each iteration of the game can be won with probability γ if and only if Alice can send qubits to Bob and Charlie at rates Q_1 and Q_2 that satisfy the constraints in Theorem 8 with respect to some correlation $P_{BC|XY} \in \mathscr{S}(\gamma)$.

VI. Two Node Analysis (Classical Links)

Consider the two node network in Figure 9. Our proof for Theorem 1 is based on quantum resolvability [33–35].

Theorem 10 (see [33–35]). Consider an ensemble, $\{p_X, \rho_A^x\}_{x\in\mathcal{X}}$, and a random codebook that consists of 2^{nR} independent sequence, $X^n(m)$, $m \in [2^{nR}]$, each is i.i.d. $\sim p_X$. If $R > I(X; A)_\rho$, then for every $\delta > 0$ and sufficiently large n,

$$\mathbb{E}\left[\left\|\rho_A^{\otimes n} - \frac{1}{2^{nR}}\sum_{m=1}^{2^{nR}} \rho_{A^n}^{X^n(m)}\right\|_1\right] \leq \delta \,, \tag{71}$$

where $\rho_{A^n}^{x^n} \equiv \bigotimes_{k=1}^n \rho_A^{x_k}$, and the expectation is over all realizations of the random codebook.

A. Achievability proof

Assume (R_0, R_1) is in the interior of $\mathcal{R}_{\text{2-node}}(\omega)$. We need to construct a code that consists of an encoding channel $F(m_1|x^n, m_0)$ and a c-q decoding channel $\mathcal{D}_{M_0 M_1 \to B^n}$, such that the error requirement in (19) holds.

By the definition of $\mathscr{S}_{\text{2-node}}(\omega)$, there exists a c-q state

$$\sigma_{UXB} = \sum_{u \in \mathcal{U}} p_U(u) |u\rangle\langle u|_U \otimes \sigma^u_{XB} \tag{72}$$

such that

$$\sigma^u_{XB} = \sum_{x \in \mathcal{X}} p_{X|U}(x|u) |x\rangle\langle x|_X \otimes \theta^u_B , \; u \in \mathcal{U} \tag{73}$$

$$\sigma_{XB} = \omega_{XB} , \tag{74}$$

$$R_1 \geq I(X;U)_\sigma \, , \, R_0 + R_1 \geq I(XB;U)_\sigma \, . \tag{75}$$

Classical codebook generation: Select a random codebook $\mathscr{C} = \{u^n(m_0, m_1)\}$ by drawing $2^{n(R_0+R_1)}$ i.i.d. sequences according to the distribution $p^n_U(u^n) = \prod_{k=1}^n p_U(u_k)$. Reveal the codebook to Alice and Bob.

Let (m_0, m_1) be a pair of random indices, uniformly distributed over $[2^{nR_0}] \times [2^{nR_1}]$. Define the following PMF

$$\widetilde{P}_{X^n M_0 M_1}(x^n, m_0, m_1) \equiv \frac{1}{2^{n(R_0+R_1)}} p^n_{X|U}(x^n | u^n(m_0, m_1)) \, . \tag{76}$$

Encoder: We define the encoding channel F as the conditional distribution above, i.e., $F = \widetilde{P}_{M_1|X^n M_0}$.

Decoder: As Bob receives m_1 from Alice, and the random element m_0, he prepares the output state $\mathcal{D}_{M_0 M_1 \to B^n}(m_0, m_1) = \theta^{u^n(m_0,m_1)}_{B^n}$.

Error analysis: Let $\delta > 0$. The encoder sends $m_1 \sim F(\cdot|x^n, m_0)$. Given $M_0 = m_0$, by the classical resolvability theorem, Cuff [26] has shown that $R_1 \geq I(X;U)_\sigma$ guarantees

$$\mathbb{E}\left\| \widetilde{P}_{M_0 X^n} - p_{M_0} \times p^n_X \right\|_1 \leq \delta \tag{77}$$

for sufficiently large n, where $\widetilde{P}_{M_0 X^n}$ is as in (76). Recall that $\widetilde{P}_{M_0 X^n}$ is random, since the codebook $\mathscr{C}$ is random. Hence, the expectation is over all realizations of $\mathscr{C}$. The resulting state is

$$\widehat{\rho}_{X^n B^n} = \frac{1}{2^{nR_0}} \sum_{m_0, x^n} \left(p^n_X(x^n) |x^n\rangle\langle x^n|_{X^n} \otimes \sum_{m_1 \in [2^{nR_1}]} \widetilde{P}_{M_1|X^n M_0}(m_1|x^n, m_0) \theta^{u^n(m_0,m_1)}_{B^n} \right) \tag{78}$$

According to (77), the probability distributions $\widetilde{P}_{M_0, X^n}$ and $p_{M_0} \times p^n_X$ are close on average. Then, let

$$\widehat{\tau}_{X^n B^n} \equiv \sum_{m_0, x^n} \widetilde{P}_{M_0 X^n}(m_0 x^n) |x^n\rangle\langle x^n|_{X^n} \otimes \sum_{m_1 \in [2^{nR_1}]} \widetilde{P}_{M_1|X^n M_0}(m_1|x^n, m_0) \theta^{u^n(m_0,m_1)}_{B^n} \, . \tag{79}$$

By (77), it follows that

$$\mathbb{E} \left\| \widehat{\tau}_{X^n B^n} - \widehat{\rho}_{X^n B^n} \right\|_1 \leq \delta \, . \tag{80}$$

Observe that

$$\begin{aligned}
\widehat{\tau}_{X^n B^n} &= \sum_{m_0, m_1, x^n} \widetilde{P}_{M_0 M_1 X^n}(m_0, m_1, x^n) |x^n\rangle\langle x^n|_{X^n} \otimes \theta^{u^n(m_0,m_1)}_{B^n} \\
&= \frac{1}{2^{n(R_0+R_1)}} \sum_{m_0, m_1, x^n} p^n_{X|U}(x^n | u^n(m_0, m_1)) |x^n\rangle\langle x^n|_{X^n} \otimes \theta^{u^n(m_0,m_1)}_{B^n} \\
&= \frac{1}{2^{n(R_0+R_1)}} \sum_{m_0, m_1} \sigma^{u^n(m_0,m_1)}_{X^n B^n}
\end{aligned} \tag{81}$$

where the second equality is due to the definition of $\widetilde{P}$ in (76), and the last line follows from (73).

Thus, according to the quantum resolvability theorem, Theorem 10, when applied to the joint system XB, for $R_0 + R_1 \geq I(XB;U)_\sigma$, we have

$$\mathbb{E}\left\| \sigma^{\otimes n}_{XB} - \widehat{\tau}_{X^n B^n} \right\|_1 \leq \delta \tag{82}$$

for sufficiently large n. Therefore, by the triangle inequality,

$$\begin{aligned}
\mathbb{E}\left\| \omega^{\otimes n}_{XB} - \widehat{\rho}_{X^n B^n} \right\|_1 &\leq \mathbb{E}\left\| \omega^{\otimes n}_{XB} - \widehat{\tau}_{X^n B^n} \right\|_1 + \mathbb{E}\left\| \widehat{\tau}_{X^n B^n} - \widehat{\rho}_{X^n B^n} \right\|_1 \\
&\leq 2\delta
\end{aligned} \tag{83}$$

by (74), (80) and (82). □

B. Converse proof

Let (R_0, R_1) be an achievable rate pair. Then, there exists a sequence $\left(2^{nR_0}, 2^{nR_1}, n\right)$ of coordination codes such that the joint quantum state $\widehat{\rho}_{X^n B^n}$ satisfies

$$\left\|\omega_{XB}^{\otimes n} - \widehat{\rho}_{X^n B^n}\right\|_1 \leq \varepsilon_n \tag{84}$$

where ε_n tends to zero as $n \to \infty$.

Fix an index $i \in \{1, \dots, n\}$. By trace monotonicity [38], taking the partial trace over X_j, B_j, $j \neq i$, maintains the inequality. Thus,

$$\|\omega_{XB} - \widehat{\rho}_{X_i B_i}\|_1 \leq \varepsilon_n \, . \tag{85}$$

Then, by the AFW inequality [46],

$$\left|H\left(X^n B^n\right)_{\widehat{\rho}} - nH\left(XB\right)_\omega\right| \leq n\beta_n \, , \tag{86}$$

and

$$\left|H\left(X_i B_i\right)_{\widehat{\rho}} - H\left(XB\right)_\omega\right| \leq \beta_n \, , \tag{87}$$

for $i \in [n]$, where β_n tends to zero as $n \to \infty$. Therefore,

$$\begin{aligned}\left|H\left(X^n B^n\right)_{\widehat{\rho}} - \sum_{i=1}^n H\left(X_i B_i\right)_{\widehat{\rho}}\right| &\leq \left|H\left(X^n B^n\right)_{\widehat{\rho}} - nH\left(XB\right)_\omega\right| + \left|nH\left(XB\right)_\omega - \sum_{i=1}^n H\left(X_i B_i\right)_{\widehat{\rho}}\right| \\ &\leq \left|H\left(X^n B^n\right)_{\widehat{\rho}} - nH\left(XB\right)_\omega\right| + \sum_{i=1}^n \left|H\left(XB\right)_\omega - H\left(X_i B_i\right)_{\widehat{\rho}}\right| \\ &\leq 2n\beta_n \, .\end{aligned} \tag{88}$$

Now, we have

$$\begin{aligned} n(R_0 + R_1) &\geq H(M_0 M_1) && (89) \\ &\geq I(X^n B^n; M_0 M_1)_{\widehat{\rho}} && (90)\end{aligned}$$

since the conditional entropy is nonnegative for classical and c-q states, and the CR element M_0 is statistically independent of the source X^n. Furthermore, by entropy sub-additivity [38],

$$\begin{aligned} I(X^n B^n; M_0 M_1)_{\widehat{\rho}} &\geq H(X^n B^n)_{\widehat{\rho}} - \sum_{i=1}^n H(X_i B_i | M_0 M_1)_{\widehat{\rho}} \\ &\geq \sum_{i=1}^n I(X_i B_i; M_0 M_1)_{\widehat{\rho}} - 2n\beta_n \end{aligned} \tag{91}$$

where the last inequality follows from (88). Defining a time-sharing variable $I \sim \text{Unif}[n]$, this can be written as

$$R_0 + R_1 + 2\beta_n \geq I(X_I B_I; M_0 M_1 | I)_{\widehat{\rho}} \tag{92}$$

with respect to the extended state:

$$\widehat{\rho}_{I M_0 M_1 X_I B_I} = \frac{1}{n} \sum_{i=1}^n |i\rangle\langle i| \otimes \widehat{\rho}_{M_0 M_1 X_i B_i} \, . \tag{93}$$

Observe that by (85) and the triangle inequality,

$$\begin{aligned} \|\omega_{XB} - \widehat{\rho}_{X_I B_I}\|_1 &= \left\|\omega_{XB} - \frac{1}{n}\sum_{i=1}^n \widehat{\rho}_{X_i B_i}\right\|_1 \\ &\leq \varepsilon_n \, . \end{aligned} \tag{94}$$

Thus, by the AFW inequality,

$$\begin{aligned} I(X_I B_I; I)_{\widehat{\rho}} &= H(X_I B_I)_{\widehat{\rho}} - \frac{1}{n}\sum_{i=1}^n H(X_i B_i)_{\widehat{\rho}} \\ &\leq \gamma_n \, , \end{aligned} \tag{95}$$

where γ_n tends to zero. Together with (92), it follows that

$$R_0 + R_1 + 2\beta_n + \gamma_n \geq I(X_I B_I; M_0 M_1 I)_{\widehat{\rho}} \tag{96}$$

By similar arguments,

$$R_1 + 2\beta_n + \gamma_n \geq I(X_I; M_0 M_1 I) \tag{97}$$

To complete the converse proof, we identify U, X, and B with (M_0, M_1, I), X_I, and B_I, respectively. Observe that given (m_0, m_1, i), the joint state of X_I and B_I is $\left(\sum_{x_i \in \mathcal{X}} p_{X_i|M_0 M_1}(x_i|m_0, m_1)\, |x_i\rangle\langle x_i|_{X_I}\right) \otimes \rho_{B_i}^{(m_0,m_1)}$, where $p_{X^n|M_0M_1}$ is the a posteriori probability distribution. Thus, there X and B are uncorrelated when conditioned on U, as required.

The bound on $|\mathcal{U}|$ follows by applying the Caratheodory theorem to the real-valued parametric representation of density matrices, as in [47, App. B]. □

VII. No-Communication Analysis

Consider the no-communication network in Figure 10, of a quantum state ω_{ABC}. To prove Theorem 4, we use similar tools. The achievability proof is straightforward, and it is thus omitted.

Then, consider the converse part. Assume that R_0 is achievable. Therefore, there exists a sequence of $(2^{nR_0}, n)$ of coordination codes such that for sufficiently large values of n,

$$\left\| \widehat{\rho}_{A^n B^n C^n} - \omega_{ABC}^{\otimes n} \right\|_1 \leq \varepsilon_n\,, \tag{98}$$

where $\varepsilon_n \to 0$ as $n \to \infty$.

Applying the chain rule,

$$\begin{aligned} nR_0 &\geq H(M_0) && (99)\\ &\geq I(A^n B^n C^n; M_0)_{\widehat{\rho}} && (100)\\ &= \sum_{i=1}^{n} I(A_i B_i C_i; M_0 | A^{i-1} B^{i-1} C^{i-1})_{\widehat{\rho}} && (101) \end{aligned}$$

For every $i \in [n]$, by trace monotonicity [38],

$$\left\| \omega_{ABC}^{\otimes i} - \widehat{\rho}_{A^i B^i C^i} \right\|_1 \leq \varepsilon_n\,. \tag{102}$$

Then, by the AFW inequality [46] [38, Ex. 11.10.2],

$$\left| I(A_i B_i C_i; A^{i-1} B^{i-1} C^{i-1})_{\widehat{\rho}} - I(A_i B_i C_i; A^{i-1} B^{i-1} C^{i-1})_{\omega^{\otimes i}} \right| \leq \beta_n\,, \tag{103}$$

where β_n tends to zero as $n \to \infty$. That is,

$$I(A_i B_i C_i; A^{i-1} B^{i-1} C^{i-1})_{\widehat{\rho}} \leq \beta_n \tag{104}$$

since $A_i B_i C_i$ and $(A_j B_j C_j)_{j<i}$ are in a product state $\omega \otimes \omega^{\otimes (k-1)}$. Hence, by (101),

$$\begin{aligned} nR_0 &\geq \sum_{i=1}^{n} I(A_i B_i C_i; M_0 A^{i-1} B^{i-1} C^{i-1})_{\widehat{\rho}} - n\beta_n \\ &\geq \sum_{i=1}^{n} I(A_i B_i C_i; M_0)_{\widehat{\rho}} - n\beta_n \\ &\geq n \left(\inf_{\sigma_{UABC} \in \mathscr{S}_{NC}(\omega)} I(U; ABC)_\sigma - 2\beta_n \right) && (105) \end{aligned}$$

taking $U \equiv M_0$, as the encoders are uncorrelated given M_0. □

VIII. Broadcast Analysis (Classical Links)

Consider coordination in broadcast network, as in Figure 11 in the main text, of a classical-quantum-quantum state ω_{XBC}. To prove the capacity theorem, Theorem 5, we use similar tools as in Section VI.

A. Achievability proof

Assume (R_0, R_1) is in the interior of $\mathcal{R}_{\text{BC}}(\omega)$. We need to construct a code that consists of an encoding channel $F(m_1|x^n, m_0)$ and a two c-q decoding channels $\mathcal{D}_{M_0M_1\to B^n}$ and $\mathcal{D}_{M_0M_1\to C^n}$,such that

$$\left\|\omega_{XB}^{\otimes n} - \frac{1}{2^{nR_0}} \sum_{m_0\in[2^{nR_0}]} \sum_{x^n\in\mathcal{X}^n} p_X^n(x^n) \left|x^n\middle\rangle\!\middle\langle x^n\right|_{X^n} \otimes \sum_{m_1\in[2^{nR_1}]} F(m_1|x^n, m_0)\mathcal{D}_{M_0M_1\to B^n}(m_1, m_0) \otimes \mathcal{D}_{M_0M_1\to C^n}(m_1, m_0)\right\|_1 \leq \varepsilon\,. \tag{106}$$

According to the definition of $\mathscr{S}_{\text{BC}}(\omega)$ (see Subsection III-C), there exists a c-q state σ_{XUBC} that can be written as

$$\sigma_{XUBC} = \sum_{(x,u)\in\mathcal{X}\times\mathcal{U}} p_{X,U}(x, u) \left|x\middle\rangle\!\middle\langle x\right|_X \otimes \left|u\middle\rangle\!\middle\langle u\right|_U \otimes \theta_B^u \otimes \eta_C^u \tag{107}$$

and satisfy

$$\sigma_{XBC} = \omega_{XBC} \tag{108}$$

We will also consider conditioning on $U = u$, and denote

$$\sigma_{XBC}^u = \sum_{x\in\mathcal{X}} p_{X|U}(x|u) \left|x\middle\rangle\!\middle\langle x\right|_X \otimes \theta_B^u \otimes \eta_C^u\,. \tag{109}$$

Classical codebook generation: Select a random codebook $\mathscr{C}_{\text{BC}} = \{u^n(m_0, m_1)\}$ by drawing $2^{n(R_0+R_1)}$ i.i.d. sequences according to the distribution p_U^n. Reveal the codebook.

Encoder: Define the encoding channel as $F = \widetilde{P}_{M_1|X^nM_0}$, where $\widetilde{P}_{X^nM_0M_1}$ be a joint distribution as in (76).

Decoders: As Bob and Charlie receive m_1 from Alice, and the random element m_0, they prepare the following output states,

$$\mathcal{D}^{(1)}_{M_0M_1\to B^n}(m_0, m_1) = \theta_B^{u^n(m_0,m_1)}\,, \tag{110}$$

$$\mathcal{D}^{(2)}_{M_0M_1\to C^n}(m_0, m_1) = \eta_C^{u^n(m_0,m_1)}\,. \tag{111}$$

Error analysis: Let $\delta > 0$. The encoder sends $m_1 \sim F(\cdot|x^n, m_0)$. As in Subsection VI-A, given m_0, if $R_1 \geq I(X;U)$, then

$$\mathbb{E}\left\|\widetilde{P}_{M_0X^n} - p_{M_0} \times p_X^n\right\|_1 \leq \delta \tag{112}$$

for sufficiently large n. As $\widetilde{P}_{M_0X^n}$ depends on the random codebook $\mathscr{C}_{\text{BC}}$, the expectation is over all realizations of $\mathscr{C}_{\text{BC}}$. The resulting state is

$$\begin{aligned}
&\widehat{\rho}_{X^nB^nC^n}\\
&= \frac{1}{2^{nR_0}} \sum_{m_0\in[2^{nR_0}]} \sum_{x^n\in\mathcal{X}^n} \Big(p_X^n(x^n) \left|x^n\middle\rangle\!\middle\langle x^n\right|_{X^n} \otimes \sum_{m_1\in[2^{nR_1}]} F(m_1|x^n, m_0)\mathcal{D}_{M_0M_1\to B^n}(m_0, m_1) \otimes \mathcal{D}_{M_0M_1\to C^n}(m_0, m_1)\Big)\\
&= \frac{1}{2^{nR_0}} \sum_{m_0,x^n} \Big(p_X^n(x^n) \left|x^n\middle\rangle\!\middle\langle x^n\right|_{X^n} \otimes \sum_{m_1\in[2^{nR_1}]} \widetilde{P}_{M_1|X^nM_0}(m_1|x^n, m_0)\theta_B^{u^n(m_0,m_1)} \otimes \eta_C^{u^n(m_0,m_1)}\Big)\,.
\end{aligned} \tag{113}$$

According to (112), the probability distributions $\widetilde{P}_{M_0X^n}$ and $p_{M_0} \times p_X^n$ are close on average. Then, let

$$\widehat{\tau}_{X^nB^nC^n} \equiv \sum_{m_0,x^n} \widetilde{P}_{M_0X^n}(m_0, x^n) \left|x^n\middle\rangle\!\middle\langle x^n\right|_{X^n} \otimes \sum_{m_1\in[2^{nR_1}]} \widetilde{P}_{M_1|X^nM_0}(m_1|x^n, m_0)\theta_B^{u^n(m_0,m_1)} \otimes \eta_C^{u^n(m_0,m_1)}\,. \tag{114}$$

Then, it follows that

$$\mathbb{E}\left\|\widehat{\tau}_{X^nB^nC^n} - \widehat{\rho}_{X^nB^nC^n}\right\|_1 \leq \delta\,, \tag{115}$$

by (112). Observe that

$$\begin{aligned}
\widehat{\tau}_{X^nB^nC^n} &= \sum_{m_0,m_1,x^n} \widetilde{P}_{M_0M_1X^n}(m_0, m_1, x^n) \left|x^n\middle\rangle\!\middle\langle x^n\right|_{X^n} \otimes \theta_B^{u^n(m_0,m_1)} \otimes \eta_C^{u^n(m_0,m_1)}\\
&= \frac{1}{2^{n(R_0+R_1)}} \sum_{m_0,m_1,x^n} p_{X|U}^n(x^n|u^n(m_0, m_1)) \left|x^n\middle\rangle\!\middle\langle x^n\right|_{X^n} \otimes \theta_B^{u^n(m_0,m_1)} \otimes \eta_C^{u^n(m_0,m_1)}\\
&= \frac{1}{2^{n(R_0+R_1)}} \sum_{m_0,m_1} \sigma_{X^nB^nC^n}^{u^n(m_0,m_1)}\,,
\end{aligned} \tag{116}$$

where the second equality is due to the definition of $\widetilde{P}$ in (76), and the last line follows from (109).

Thus, according to the quantum resolvability theorem 10, when applied to the joint system XBC, we have

$$\mathbb{E}\left\|\sigma_{XB^nC^n}^{\otimes n}-\widehat{\tau}_{X^nB^nC^n}\right\|_1\leq\delta \tag{117}$$

for sufficiently large n. Therefore, by the triangle inequality,

$$\begin{aligned}\mathbb{E}\left\|\omega_{XBC}^{\otimes n}-\widehat{\rho}_{X^nB^nC^n}\right\|_1&\leq\mathbb{E}\left\|\omega_{XB^nC^n}^{\otimes n}-\widehat{\tau}_{X^nB^nC^n}\right\|_1+\mathbb{E}\|\widehat{\tau}_{X^nB^nC^n}-\widehat{\rho}_{X^nB^nC^n}\|_1\\&\leq 2\delta\end{aligned} \tag{118}$$

by (108), (115) and (117). □

B. Converse proof

Let (R_0,R_1) be an achievable coordination rate pair for the simulation of a c-q-q state ω_{XBC} in the broadcast setting. Then, there exists a sequence of $\left(2^{nR_0},2^{nR_1},n\right)$ coordination codes such that the joint quantum state $\widehat{\rho}_{X^nB^nC^n}$ satisfies

$$\left\|\omega_{XB^nC^n}^{\otimes n}-\widehat{\rho}_{X^nB^nC^n}\right\|_1\leq\varepsilon_n\,, \tag{119}$$

where ε_n tends to zero as $n\to\infty$. Fix an index $i\in\{1,\ldots,n\}$. By trace monotonicity [38], taking the partial trace over X_j, B_j, C_j, for $j\neq i$, maintains the inequality, thus Thus,

$$\|\omega_{XBC}-\widehat{\rho}_{X_iB_iC_i}\|_1\leq\varepsilon_n\,. \tag{120}$$

Then, by the AFW inequality [46],

$$\left|H\left(X^nB^nC^n\right)_{\widehat{\rho}}-nH\left(XBC\right)_\omega\right|\leq n\beta_n\,, \tag{121}$$

and

$$\left|H\left(X_iB_iC_i\right)_{\widehat{\rho}}-H\left(XBC\right)_\omega\right|\leq\beta_n\,, \tag{122}$$

for $i\in[n]$, where β_n tends to zero as $n\to\infty$. Therefore,

$$\left|H\left(X^nB^nC^n\right)_{\widehat{\rho}}-\sum_{i=1}^nH\left(X_iB_iC_i\right)_{\widehat{\rho}}\right|\leq 2n\beta_n\,. \tag{123}$$

Now, we also have

$$\begin{aligned}n(R_0+R_1)&\geq H(IJ)\\&\geq I(X^nB^nC^n;IJ)_{\widehat{\rho}}\,,\end{aligned} \tag{124}$$

since the conditional entropy is nonnegative for classical and c-q-q states, and the CR element J is statistically independent of the source X^n. Furthermore, by entropy sub-additivity [38],

$$\begin{aligned}I(X^nB^nC^n;M_0M_1)_{\widehat{\rho}}&\geq H(X^nB^nC^n)_{\widehat{\rho}}-\sum_{i=1}^nH(X_iB_iC_i|M_0M_1)_{\widehat{\rho}}\\&\geq\sum_{i=1}^nI(X_iB_iC_i;M_0M_1)_{\widehat{\rho}}-2n\beta_n\end{aligned} \tag{125}$$

where the last inequality follows from (123).

Defining a time-sharing variable $I\sim\text{Unif}[n]$, this can be written as

$$R_0+R_1+2\beta_n\geq I(X_IB_IC_I;M_0M_1|I)_{\widehat{\rho}} \tag{126}$$

with respect to the extended state

$$\widehat{\rho}_{IM_0M_1X_IB_IC_I}=\frac{1}{n}\sum_{i=1}^n|i\rangle\langle i|\otimes\widehat{\rho}_{M_0M_1X_iB_iC_i}\,. \tag{127}$$

Observe that

$$\|\omega_{XBC}-\widehat{\rho}_{X_IB_IC_I}\|_1=\left\|\omega_{XBC}-\frac{1}{n}\sum_{i=1}^n\widehat{\rho}_{X_iB_iC_i}\right\|_1\leq\varepsilon_n \tag{128}$$

by the triangle inequality (see (120)). Thus, by the AFW inequality,

$$I(X_I B_I C_I; I)_{\widehat{\rho}} = H(X_I B_I C_I)_{\widehat{\rho}} - \frac{1}{n}\sum_{i=1}^{n} H(X_i B_i C_i)_{\widehat{\rho}}$$
$$\leq \gamma_n \,, \tag{129}$$

where γ_n tends to zero as $n \to \infty$. Together with (125), it implies

$$R_0 + R_1 + 2\beta_n + \gamma_n \geq I(X_I B_I C_I; M_0 M_1 I)_{\widehat{\rho}} \,. \tag{130}$$

By similar arguments,

$$R_1 + 2\beta_n + \gamma_n \geq I(X_I; M_0 M_1) \,. \tag{131}$$

To complete the converse proof, we identify U, X, and BC with (M_0, M_1, I), X_I, and $B_I C_I$, respectively. Observe that given (m_0, m_1, i), the joint state of X_I, B_I and C_I is

$$\left(\sum_{x_i \in \mathcal{X}} p_{X_i|M_0 M_1}(x_i|m_0, m_1)\, |x_i\rangle\langle x_i|_{X_I} \right) \otimes \rho_{B_I}^{(m_0,m_1)} \otimes \rho_{C_I}^{(m_0,m_1)} \,, \tag{132}$$

where $p_{X^n|M_0 M_1}$ is the a posteriori probability distribution. Thus, there is no correlation between X, B, and C when conditioned on U, as required. □

IX. Cascade Network Analysis (Quantum Links)

We prove the rate characterization in Theorem 6. Consider the cascade network in Figure 12.

A. Achievability proof

The proof for the direct part exploits the state redistribution result by Yard and Devetak in [12]. We first describe the state redistribution problem. Consider two parties, Alice and Bob. Let their systems be described by the joint state ψ_{ABG}, where A and B belong to Alice, and G belongs to Bob. Let the state $|\psi_{ABGR}\rangle$ be a purification of ψ_{ABG}. Alice and Bob would like to redistribute the state ψ_{ABG} such that B is transferred from Alice to Bob. Alice can send quantum description systems at rate Q and they share maximally entangled pairs of qubits at a rate E.

Theorem 11 (State Redistribution [12]). The optimal rates for state redistribution of $|\psi_{AGBR}\rangle$ with rate-limited entanglement are

$$Q \geq \frac{1}{2} I(B; R|G)_\psi \,, \tag{133}$$
$$Q + E \geq H(B|G)_\psi \,. \tag{134}$$

We go back to the coordination setting for the cascade network (see Figure 12). Alice, Bob, and Charlie would like to simulate the joint state $\omega_{ABC}^{\otimes n}$. Let $|\omega_{ABCR}\rangle$ be a purification. Suppose that Alice prepares the desired state $|\omega_{A\bar{B}\bar{C}\bar{R}}\rangle^{\otimes n}$ locally in her lab, where $\bar{B}^n$, $\bar{C}^n$, and $\bar{R}^n$ are her ancillas. Let $\varepsilon > 0$ be arbitrarily small. By the state redistribution theorem, Theorem 11, Alice can transmit $\bar{B}^n \bar{C}^n$ to Bob at communication rate Q_1 and entanglement rate E_1, provided that

$$Q_1 \geq \frac{1}{2} I(\bar{B}\bar{C}; \bar{R})_\omega = \frac{1}{2} I(BC; R)_\omega \,, \tag{135}$$
$$Q_1 + E_1 \geq H(\bar{B}\bar{C})_\omega = H(BC)_\omega \tag{136}$$

(see [12]). That is, there exist a bipartite state $\Psi_{T_A T_B'}$ and encoding maps, $\mathcal{E}^{(1)}_{\bar{B}^n \bar{C}^n T_A \to M_1}$ and $\mathcal{F}^{(1)}_{M_1 T_B' \to B^n \widetilde{C}^n}$, such that

$$\left\| \tau^{(1)}_{\bar{R}^n A^n B^n \widetilde{C}^n} - \omega^{\otimes n}_{RABC} \right\|_1 \leq \varepsilon, \tag{137}$$

for sufficiently large n, where

$$\tau^{(1)}_{\bar{R}^n A^n B^n \widetilde{C}^n} = \left[\mathrm{id}_{\bar{R}^n A^n} \otimes \mathcal{F}^{(1)}_{M_1 T_B' \to B^n \widetilde{C}^n} \circ \left(\mathcal{E}^{(1)}_{\bar{B}^n \bar{C}^n T_A \to M_1} \otimes \mathrm{id}_{T_B'} \right) \right] \left(\omega^{\otimes n}_{RABC} \otimes \Psi_{T_A T_B'} \right) \,. \tag{138}$$

Similarly, $\bar{C}^n$ can be compressed and transmitted with rates

$$Q_2 \geq \frac{1}{2} I(\bar{C}; A\bar{R})_\omega = \frac{1}{2} I(C; AR)_\omega \,, \tag{139}$$
$$Q_2 + E_2 \geq H(\bar{C})_\omega = H(C)_\omega \,, \tag{140}$$

by Theorem 11. Namely, there exists a bipartite state $\Theta_{T''_B T_C}$ and encoding maps, $\mathcal{F}^{(2)}_{\bar{C}^n T''_B\to M_2}$ and $\mathcal{D}^{(2)}_{M_2 T_C\to C^n}$, such that

$$\left\|\tau^{(2)}_{\bar{R}^n A^n B^n C^n} - \omega^{\otimes n}_{\bar{R}ABC}\right\|_1 \leq \varepsilon, \tag{141}$$

where

$$\tau^{(2)}_{\bar{R}^n A^n B^n C^n} = \left[\left(\mathrm{id}_{\bar{R}^n A^n \bar{B}^n}\otimes \mathcal{D}^{(2)}_{M_2 T_C\to C^n}\right)\circ\left(\mathcal{F}^{(2)}_{\bar{C}^n T''_B\to M_2}\otimes \mathrm{id}_{T_C}\right)\right]\left(\omega^{\otimes n}_{\bar{R}A\bar{B}\bar{C}}\otimes \Theta_{T''_B T_C}\right) . \tag{142}$$

The coding operations for the cascade network are described below.

Encoding:

A) Alice prepares $|\omega_{A\bar{B}\bar{C}\bar{R}}\rangle^{\otimes n}$ locally. She applies $\mathrm{id}_{\bar{R}^n A^n}\otimes \mathcal{E}^{(1)}_{\bar{B}^n\bar{C}^n T_A\to M_1}$, and sends M_1 to Bob.

B) As Bob receives M_1, he applies

$$\mathcal{F}_{M_1 T'_B T''_B\to B^n M_2} \equiv \left(\mathrm{id}_{B^n}\otimes \mathcal{F}^{(2)}_{\widetilde{C}^n T''_B\to M_2}\right)\circ \mathcal{F}^{(1)}_{M_1 T'_B\to B^n\widetilde{C}^n} . \tag{143}$$

C) Charlie receives M_2 from Bob and applies $\mathcal{D}^{(2)}_{M_2 T_C\to C^n}$.

Error analysis: We trace out the reference system R and write the analysis with respect to the reduced states. The joint state after Alice's encoding is

$$\rho^{(1)}_{A^n M_1 T'_B} = \left[\mathrm{id}_{A^n}\otimes \mathcal{E}^{(1)}_{\bar{B}^n\bar{C}^n T_A\to M_1}\otimes \mathrm{id}_{T'_B}\right](\omega^{\otimes n}_{A\bar{B}\bar{C}}\otimes \Psi_{T_A T'_B}) . \tag{144}$$

After Bob applies his encoder, this results in

$$\begin{aligned}\rho^{(2)}_{A^n B^n M_2 T_C} &= \left[\left(\mathrm{id}_{A^n B^n}\otimes \mathcal{F}^{(2)}_{\widetilde{C}^n T''_B\to M_2}\otimes \mathrm{id}_{T_C}\right)\circ\left(\mathrm{id}_{A^n}\otimes \mathcal{F}^{(1)}_{M_1 T'_B\to B^n\widetilde{C}^n}\otimes \mathrm{id}_{T''_B T_C}\right)\right](\rho^{(1)}_{A^n M_1 T'_B}\otimes \Theta_{T''_B T_C}) \\ &= \left(\mathrm{id}_{A^n B^n}\otimes \mathcal{F}^{(2)}_{\widetilde{C}^n T''_B\to M_2}\otimes \mathrm{id}_{T_C}\right)(\tau^{(1)}_{A^n B^n\widetilde{C}^n}\otimes \Theta_{T''_B T_C})\end{aligned} \tag{145}$$

by (143), and based on the definition of $\tau^{(1)}$ in (138). According to (137), $\tau^{(1)}$ and $\omega^{\otimes n}$ are close in trace distance. By trace monotonicity under quantum channels, we have

$$\left\|\rho^{(2)}_{A^n B^n M_2 T_C} - \left(\mathrm{id}_{A^n B^n}\otimes \mathcal{F}^{(2)}_{\widetilde{C}^n T''_B\to M_2}\otimes \mathrm{id}_{T_C}\right)(\omega^{\otimes n}_{AB\widetilde{C}}\otimes \Theta_{T''_B T_C})\right\|_1 \leq \varepsilon. \tag{146}$$

As Charlie receives M_2 and encodes, the final state, at the output of the cascade network, is given by

$$\widehat{\rho}_{A^n B^n C^n} = \left[\mathrm{id}_{A^n B^n}\otimes \mathcal{D}^{(2)}_{M_2 T_C\to C^n}\right](\rho^{(2)}_{A^n B^n M_2 T_C}) . \tag{147}$$

Once more, by trace monotonicity,

$$\left\|\widehat{\rho}_{A^n B^n C^n} - \tau^{(2)}_{A^n B^n C^n}\right\|_1 \leq \varepsilon . \tag{148}$$

(see (141) and (142)). Thus, using (141), (148), and the triangle inequality, we have

$$\begin{aligned}\left\|\widehat{\rho}_{A^n B^n C^n} - \omega^{\otimes n}_{ABC}\right\|_1 &\leq \left\|\tau^{(2)}_{A^n B^n C^n} - \omega^{\otimes n}_{ABC}\right\|_1 + \left\|\widehat{\rho}_{A^n B^n C^n} - \tau^{(2)}_{A^n B^n C^n}\right\|_1 \\ &\leq 2\varepsilon .\end{aligned} \tag{149}$$

This completes the achievability proof for the cascade network.

B. Converse proof

We now prove the converse part for Theorem 6. Recall that in the cascade network, each party shares entanglement with their nearest neighbor a priori, i.e., Alice and Bob share $\left|\Psi_{T_A T'_B}\right\rangle$, while Bob and Charlie share $\left|\Theta_{T''_B T_C}\right\rangle$ (see Figure 12 in subsection IV-A). Alice applies an encoding map $\mathcal{E}_{\bar{A}^n T_A\to A^n M_1}$ on her part, and sends the output M_1 to Bob. As Bob receives M_1, he encodes using a map $\mathcal{F}_{M_1 T'_B T''_B\to B^n M_2}$, and sends M_2. As Charlie receives M_2, he applies an encoding channel $\mathcal{D}_{M_2 T_C\to C^n}$. Suppose that Alice prepares the state $|\omega_{R\bar{A}\bar{B}\bar{C}}\rangle^{\otimes n}$ locally, and then encodes as explained above. The protocol can be described through the following relations:

$$\rho^{(1)}_{R^n A^n M_1 T'_B} = (\mathrm{id}_{R^n}\otimes \mathcal{E}_{\bar{A}^n T_A\to A^n M_1}\otimes \mathrm{id}_{T'_B})(\omega^{\otimes n}_{R\bar{A}}\otimes \Psi_{T_A T'_B}) , \tag{150}$$

$$\rho^{(2)}_{R^n A^n B^n M_2 T_C} = (\mathrm{id}_{\bar{R}^n A^n}\otimes \mathcal{F}_{M_1 T'_B T''_B\to B^n M_2}\otimes \mathrm{id}_{T_C})(\rho^{(1)}_{R^n A^n M_1 T'_B}\otimes \Theta_{T''_B T_C}) , \tag{151}$$

$$\widehat{\rho}_{R^n A^n B^n C^n} = (\mathrm{id}_{R^n A^n B^n}\otimes \mathcal{D}_{M_2 T_C\to C^n})\left(\rho^{(2)}_{R^n A^n B^n M_2 T_C}\right) . \tag{152}$$

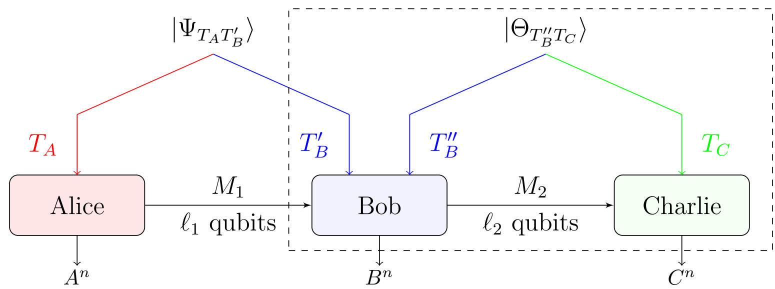


Fig. 16: At first, we treat the encoding operation of Bob and Charlie as a black box.

Let (Q_1, Q_2, E_1, E_2) be achievable rate tuple for coordination in the cascade network. Then, there exists a sequence of codes such that

$$\left\|\widehat{\rho}_{R^n A^n B^n C^n} - \omega_{RABC}^{\otimes n}\right\|_1 \leq \varepsilon_n \tag{153}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Consider Alice's communication and entanglement rates, Q_1 and E_1. At this point, we may view the entire encoding operation of Bob and Charlie as a black box whose input and output are (M_1, T_B') and (B^n, C^n), respectively, as illustrated in Figure 16. Now,

$$\begin{aligned} 2n(Q_1 + E_1) &= 2\left[\log\dim(\mathcal{H}_{M_1}) + \log\dim(\mathcal{H}_{T_B'})\right] \\ &\geq I(M_1 T_B'; A^n R^n)_{\rho^{(1)}} \end{aligned} \tag{154}$$

since the quantum mutual information satisfies $I(A;B)_\rho \leq 2\log\dim(\mathcal{H}_A)$ in general. Therefore, by the data processing inequality,

$$\begin{aligned} I(M_1 T_B'; A^n R^n)_{\rho^{(1)}} &\geq I(B^n C^n; A^n R^n)_{\widehat{\rho}} \\ &\geq I(B^n C^n; A^n R^n)_{\omega^{\otimes n}} - n\alpha_n \\ &= n[I(BC; AR)_\omega - \alpha_n], \end{aligned} \tag{155}$$

where $\alpha_n \to 0$ when $n \to \infty$. The second inequality follows from (153) and the Alicki-Fannes-Winter (AFW) inequality [46] (entropy continuity). Since $|\omega_{RABC}\rangle$ is pure, we have $I(BC; AR)_\omega = 2H(BC)_\omega$ [38, Th. 11.2.1]. Therefore, combining (154)-(155), we have

$$Q_1 + E_1 \geq H(BC)_\omega - \frac{1}{2}\alpha_n . \tag{156}$$

To show the bound on Q_1, observe that a lower bound on the communication rate with unlimited entanglement resources also holds with limited resources. Therefore, the bound $Q_1 \geq \frac{1}{2} I(BC; R)_\omega$ follows from the entanglement-assisted capacity theorem due to Bennett et al. [25]. It is easier to see this through resource inequalities, following the arguments in [27]. If the entanglement resources are unlimited, then the coordination code achieves

$$\begin{aligned} Q_1\, [q \to q]_{A\to BC} &\geq \langle \omega_{RBC} \rangle \\ &\equiv \langle \mathrm{Tr}_A : \omega_{RABC} \rangle \\ &\geq \frac{1}{2} I(BC; R)_\omega\, [q \to q]_{A\to BC} \end{aligned} \tag{157}$$

where the resource units $[q \to q]$, $[qq]$, and $\langle \omega_{ABC} \rangle$ represent a single use of a noiseless qubit channel, an EPR pair, and the desired state ω_{ABC}, respectively, while the unit resource $\langle \mathcal{N}_{A\to B} : \rho \rangle$ indicates a simulation of the channel output from $\mathcal{N}_{A\to B}$ with respect to the input state ρ. The last inequality holds by [25, 27].

Similarly, we bound Bob's communication and entanglement rates as follows,

$$2n(Q_2 + E_2) = 2\left[\log\dim(\mathcal{H}_{M_2}) + \log\dim(\mathcal{H}_{T_B''})\right] \tag{158}$$
$$\geq I(M_2 T_C; A^n B^n R^n)_{\rho^{(2)}} \tag{159}$$
$$\geq I(C^n; A^n B^n R^n T_B'')_{\widehat{\rho}} \tag{160}$$
$$\geq n[I(C; ABR)_\omega - \beta_n] \tag{161}$$
$$= n[2H(C)_\omega - \beta_n] \tag{162}$$

where $\beta_n \to 0$ when $n \to \infty$. As before, the last inequality follows from (153) and the AFW inequality [46]. Hence,

$$Q_2 + E_2 \geq H(C)_\omega - \frac{1}{2}\beta_n \,. \tag{163}$$

Furthermore,

$$\begin{aligned} Q_2 \, [q \to q]_{B\to C} &\geq \langle \omega_{RAC} \rangle \\ &\equiv \langle \mathrm{Tr}_B : \omega_{RABC} \rangle \\ &\geq \frac{1}{2} I(C;AR)_\omega \, [q \to q]_{B\to C} \end{aligned} \tag{164}$$

which implies $Q_2 \geq \frac{1}{2} I(C;AR)_\omega$.

This completes the proof of Theorem 6 for the cascade network.

X. Broadcast Analysis (Quantum Links)

We prove the rate characterization in Theorem 8. Consider the broadcast network in Figure 14. We show achievability by using a quantum version of the binning technique.

Let $\varepsilon_i, \delta > 0$ be arbitrarily small. Define the average states,

$$\sigma_{AB}^{(x)} = \sum_{y\in\mathcal{Y}} p_{Y|X}(y|x) \sigma_{AB}^{(x,y)} \,, \tag{165}$$

$$\sigma_{AC}^{(y)} = \sum_{x\in\mathcal{X}} p_{X|Y}(x|y) \sigma_{AC}^{(x,y)} \,, \tag{166}$$

and consider a spectral decomposition of the reduced states of Bob and Charlie,

$$\sigma_B^{(x)} = \sum_{z\in\mathcal{Z}} p_{Z|X}(z|x) \, |\psi_{x,z}\rangle\!\langle\psi_{x,z}| \,, \tag{167}$$

$$\sigma_C^{(y)} = \sum_{w\in\mathcal{W}} p_{W|Y}(w|y) \, |\phi_{y,w}\rangle\!\langle\phi_{y,w}| \,, \tag{168}$$

where $p_{Z|X}$ and $p_{W|Y}$ are conditional probability distributions, and $\{|\psi_{x,z}\rangle\}_z, \{|\phi_{y,w}\rangle\}_w$ are orthonormal bases for $\mathcal{H}_B$, $\mathcal{H}_C$, respectively, for every given $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. We can also assume that the different bases are orthogonal to each other by requiring that Bob and Charlie encode on a different Hilbert space for every value of (x, y).

We use the type class definitions and notations in [38, Chap. 14]. In particular, $T_\delta^{X^n}$ denotes the δ-typical set with respect to p_X, and $T_\delta^{Z^n|x^n}$ is the conditional δ-typical set with respect to p_{XZ}, given $x^n \in T_\delta^{X^n}$.

Classical Codebook Generation: For every sequence $z^n \in \mathcal{Z}^n$, assign an index $m_1(z^n)$, uniformly at random from $[2^{nQ_1}]$. A bin $\mathfrak{B}_1(m_1)$ is defined as the subset of sequences in $\mathcal{Z}^n$ that are assigned the same index m_1, for $m_1 \in [2^{nQ_1}]$. The codebook is revealed to all parties.

Encoding:

A) Alice prepares $\omega_{A\bar{B}\bar{C}}^{\otimes n}$ locally, where $\bar{B}^n\bar{C}^n$ are her ancillas, without any correlation with X^n and Y^n (see Remark 6). She applies the encoding channel $\mathcal{E}^{(1)}_{\bar{B}^n\to M_1} \otimes \mathcal{E}^{(2)}_{\bar{C}^n\to M_2}$,

$$\mathcal{E}^{(1)}_{\bar{B}^n\to M_1}(\rho_1) = \sum_{x^n\in\mathcal{X}^n} p_X^{\otimes n}(x^n) \sum_{z^n\in\mathcal{Z}^n} \langle\psi_{x^n,z^n}|\, \rho_1 \,|\psi_{x^n,z^n}\rangle \, |m_1(z^n)\rangle\!\langle m_1(z^n)| \,, \tag{169}$$

$$\mathcal{E}^{(2)}_{\bar{C}^n\to M_2}(\rho_2) = \sum_{y^n\in\mathcal{Y}^n} p_Y^{\otimes n}(y^n) \sum_{w^n\in\mathcal{W}^n} \langle\phi_{y^n,w^n}|\, \rho_2 \,|\phi_{y^n,w^n}\rangle \, |m_2(w^n)\rangle\!\langle m_2(w^n)| \,, \tag{170}$$

for $\rho_1 \in \Delta(\mathcal{H}_B^{\otimes n})$, $\rho_2 \in \Delta(\mathcal{H}_C^{\otimes n})$, and transmits M_1 and M_2 to Bob and Charlie, respectively.

B) First, Bob applies the following encoding channel,

$$\mathcal{F}^{(x^n)}_{M_1\to B^n}(\rho_{M_1}) = \sum_{m_1=1}^{2^{nQ_1}} \langle m_1|\, \rho_{M_1} \,|m_1\rangle \left(\frac{1}{\left|T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1)\right|} \sum_{z^n\in T_\delta^{Z^n|x^n}\cap\mathfrak{B}_1(m_1)} |\psi_{x^n,z^n}\rangle\!\langle\psi_{x^n,z^n}| \right) \tag{171}$$

C) Charlie encodes in a similar manner.

Error analysis: Due to the code construction, it suffices to consider the individual errors of Bob and Charlie,

$$\frac{1}{2}\left\|\omega_{XAB}^{\otimes n}-\left(\mathcal{F}_{X^nM_1\to X^nB^n}\circ\mathcal{E}^{(1)}_{\bar{B}^n\to M_1}\right)(\omega_X^{\otimes n}\otimes\omega_{A\bar{B}}^{\otimes n})\right\|_1 , \tag{172}$$

$$\frac{1}{2}\left\|\omega_{YAC}^{\otimes n}-\left(\mathcal{D}_{Y^nM_2\to Y^nC^n}\circ\mathcal{E}^{(2)}_{\bar{C}^n\to M_2}\right)(\omega_Y^{\otimes n}\otimes\omega_{A\bar{C}}^{\otimes n})\right\|_1 , \tag{173}$$

respectively, where we use the short notation $\mathcal{E}^{(1)}_{\bar{B}^n\to M_1}\equiv \mathrm{id}_{X^nA^n}\otimes\mathcal{E}^{(1)}_{\bar{B}^n\to M_1}$, and similarly for the other encoding maps.

We now focus on Bob's error. Consider a given codebook $\mathscr{C}_1=\{m_1(z^n)\}$. Alice encodes M_1 by

$$\mathcal{E}^{(1)}_{\bar{B}^n\to M_1}(\omega_{AB}^{\otimes n})=\sum_{\tilde{x}^n\in\mathcal{X}^n}p_X^{\otimes n}(\tilde{x}^n)\sum_{z^n\in\mathcal{Z}^n}\langle\psi_{\tilde{x}^n,z^n}|\,\omega_{AB}^{\otimes n}\,|\psi_{\tilde{x}^n,z^n}\rangle\,|m_1(z^n)\rangle\!\langle m_1(z^n)| , \tag{174}$$

where we use the short notation $|\psi\rangle_{x^n,z^n}\equiv\bigotimes_{i=1}^n|\psi\rangle_{x_i,z_i}$. By the weak law of large numbers, this state is ε_1-close in trace distance to

$$\begin{aligned}\rho^{(1)}_{A^nM_1}&=\sum_{\tilde{x}^n\in T_\delta^{X^n}}p_X^{\otimes n}(\tilde{x}^n)\sum_{z^n\in T_\delta^{Z^n|\tilde{x}^n}}\langle\psi_{\tilde{x}^n,z^n}|\,\sigma^{(\tilde{x}^n)}_{A^n\bar{B}^n}\,|\psi_{\tilde{x}^n,z^n}\rangle\,|m_1(z^n)\rangle\!\langle m_1(z^n)|\\&=\sum_{x^n\in T_\delta^{X^n}}p_X^{\otimes n}(x^n)\rho^{(1|x^n)}_{A^nM_1} ,\end{aligned} \tag{175}$$

for sufficiently large n, where we have defined

$$\rho^{(1|x^n)}_{A^nM_1}=\sum_{z^n\in T_\delta^{Z^n|x^n}}\langle\psi_{x^n,z^n}|\,\sigma^{(x^n)}_{A^n\bar{B}^n}\,|\psi_{x^n,z^n}\rangle\,|m_1(z^n)\rangle\!\langle m_1(z^n)| . \tag{176}$$

Let $x^n\in T_\delta^{X^n}$. After Bob encodes B^n, we have

$$\mathcal{F}^{(x^n)}_{M_1\to B^n}\left(\rho^{(1|x^n)}_{A^nM_1}\right)=\sum_{z^n\in T_\delta^{Z^n|x^n}}\langle\psi_{x^n,z^n}|\,\sigma^{(x^n)}_{A^n\bar{B}^n}\,|\psi_{x^n,z^n}\rangle\,\mathcal{F}^{(x^n)}_{M_1\to B^n}(|m_1(z^n)\rangle\!\langle m_1(z^n)|) . \tag{177}$$

By the definition of Bob's encoding channel, $\mathcal{F}^{(x^n)}_{M_1\to B^n}$, in (171),

$$\mathcal{F}^{(x^n)}_{M_1\to B^n}(|m_1(z^n)\rangle\!\langle m_1(z^n)|)=\frac{1}{\left|T_\delta^{Z^n|x^n}\cap\mathfrak{B}_1(m_1(z^n))\right|}\sum_{\tilde{z}\in T_\delta^{Z^n|x^n}\cap\mathfrak{B}_1(m_1(z^n))}|\psi_{x^n,\tilde{z}}\rangle\!\langle\psi_{x^n,\tilde{z}}| . \tag{178}$$

Substituting in (177) yields

$$\begin{aligned}\mathcal{F}^{(x^n)}_{M_1\to B^n}\left(\rho^{(1|x^n)}_{A^nM_1}\right)=&\sum_{z^n\in T_\delta^{Z^n|x^n}}\langle\psi_{x^n,z^n}|\,\sigma^{(x^n)}_{A^n\bar{B}^n}\,|\psi_{x^n,z^n}\rangle\\&\otimes\left[\frac{1}{\left|T_\delta^{Z^n|x^n}\cap\mathfrak{B}_1(m_1(z^n))\right|}\sum_{\tilde{z}^n\in T_\delta^{Z^n|x^n}\cap\mathfrak{B}_1(m_1(z^n))}|\psi_{x^n,\tilde{z}^n}\rangle\!\langle\psi_{x^n,\tilde{z}^n}|\right] .\end{aligned} \tag{179}$$

Based on the classical result [48, Chapter 10.3], the random codebook $\mathscr{C}_1$ satisfies that

$$\Pr_{\mathscr{C}_1}\left(\exists\tilde{z}^n\in T_\delta^{Z^n|x^n}\cap\mathfrak{B}_1(m_1(z^n)):\tilde{z}^n\neq z^n\right)\leq\varepsilon_2 \tag{180}$$

given $z^n\in T_\delta^{Z^n|x^n}$, for sufficiently large n, provided that the codebook size is at least $2^{n(H(Z|X)+\varepsilon_3)}$, where $H(Z|X)$ denotes the classical conditional entropy. As $|\mathscr{C}_1|=2^{nQ_1}$, this holds if

$$\begin{aligned}Q_1&>H(Z|X)+\varepsilon_3\\&=H(B|X)_\omega+\varepsilon_3 .\end{aligned} \tag{181}$$

Observe that if the summation set in (179), $T_\delta^{Z^n|x^n}\cap\mathfrak{B}_1(m_1(z^n))$, consists of the sequence z^n alone, then the overall state in (179) is identical to the post-measurement state after a typical subspace measurement on B^n, with respect to the conditional δ-typical set $T_\delta^{Z^n|x^n}$. Based on the gentle measurement lemma [49], this state is ε_4-close to $\sigma^{(x^n)}_{AB}$, for sufficiently large n.

Therefore, by the triangle inequality and total expectation formula,

$$\begin{aligned}&\left\|\omega_{XAB}^{\otimes n}-\mathbb{E}_{\mathscr{C}_1}\left(\mathcal{F}_{X^nM_1\to X^nB^n}\circ\mathcal{E}^{(1)}_{\bar{B}^n\to M_1}\right)(\omega_X^{\otimes n}\otimes\omega_{A\bar{B}}^{\otimes n})\right\|_1\\&\leq\sum_{x^n\in\mathcal{X}^n}p_X^{\otimes n}(x^n)\cdot\mathbb{E}_{\mathscr{C}_1}\left\|\sigma^{(x^n)}_{A^nB^n}-\left(\mathcal{F}^{(x^n)}_{M_1\to B^n}\circ\mathcal{E}^{(1)}_{\bar{B}^n\to M_1}\right)\left(\sigma^{(x^n)}_{A^nB^n}\right)\right\|_1\end{aligned}$$

$$\leq \varepsilon_1 + \varepsilon_2 + \varepsilon_4 \,. \tag{182}$$

By symmetry, Charlie's error tends to zero as well, provided that $Q_2 \geq H(C|Y)_\omega + \varepsilon_5$. Since the total error vanishes, when averaged over the class of binning codebooks, it follows that there exists a deterministic codebook with the same property. The achievability proof follows by taking $n \to \infty$ and then $\varepsilon_j, \delta \to 0$.

The converse proof follows the lines of [12], and it is thus omitted. This completes the proof of Theorem 8 for the broadcast network.

XI. Multiple-Access Analysis (Quantum Links)

We prove the rate characterization in Theorem 9. Consider the multiple-access network in Figure 15. As explained in Remark 7, coordination in the multiple-access network is only possible if there exists an isometry $V : \mathcal{H}_C \to \mathcal{H}_{C_1} \otimes \mathcal{H}_{C_2}$ such that

$$(\mathbb{1} \otimes V) |\omega_{ABC}\rangle = |\phi_{AC_1}\rangle \otimes |\chi_{BC_2}\rangle \tag{183}$$

where $|\phi_{AC_1}\rangle$ and $|\chi_{BC_2}\rangle$ are purifications of ω_A and ω_B, respectively. For this reason, Theorem 9 assumes that this property holds. Furthermore, since $|\phi_{AC_1}\rangle$ and $|\chi_{BC_2}\rangle$ purify ω_A and ω_B, respectively, we have $H(C_1)_\phi = H(A)_\phi = H(A)_\omega$ and $H(C_2)_\chi = H(B)_\chi = H(B)_\omega$. Thus, it suffices to show that (Q_1, Q_2) is achievable if and only if

$$Q_1 \geq H(C_1)_\phi \,, \tag{184}$$
$$Q_2 \geq H(C_2)_\chi \,. \tag{185}$$

The achievability proof follows from the Schumacher compression protocol [50] [38, chap. 18] in a straightforward manner. Alice and Bob prepare $\phi_{AC_1}^{\otimes n}$ and $\chi_{BC_2}^{\otimes n}$, respectively. Then, they send C_1^n and C_2^n using the Schumacher compression protocol, and finally, Charlie applies the isometry $(V^\dagger)^{\otimes n}$ in order to simulate $\omega_{ABC}^{\otimes n}$ (see (183)). The details are omitted.

It remains to show the converse part. Recall that in the multiple-access network, Alice and Bob each applies their respective encoding map, $\mathcal{E}_{A^n \to A^n M_1}$ and $\mathcal{F}_{B^n \to B^n M_2}$, and send the quantum descriptions M_1 and M_2. Then, Charlie encodes by $\mathcal{D}_{M_1 M_2 \to C^n}$.

The protocol can be described through the following relations:

$$\rho^{(1)}_{A^n M_1} = \mathcal{E}_{A^n \to A^n M_1}(\omega_A^{\otimes n}) \,,\; \rho^{(2)}_{B^n M_2} = \mathcal{F}_{B^n \to B^n M_2}(\omega_B^{\otimes n}) \,, \tag{186}$$
$$\widehat{\rho}_{A^n B^n C^n} = (\mathrm{id}_{A^n B^n} \otimes \mathcal{D}_{M_1 M_2 \to C^n}) \, (\rho^{(1)}_{A^n M_1} \otimes \rho^{(2)}_{B^n M_2}) \,. \tag{187}$$

Let (Q_1, Q_2) be an achievable rate pair for coordination in the multiple-access network in Figure 15. Then, there exists a sequence of $(2^{nQ_1}, 2^{nQ_2}, n)$ coordination codes such that

$$\left\| \widehat{\rho}_{A^n B^n C^n} - \omega_{ABC}^{\otimes n} \right\|_1 \leq \varepsilon_n \tag{188}$$

tends to zero as $n \to \infty$. Applying the isometry $V^{\otimes n}$ yields

$$\left\| \widehat{\sigma}_{A^n B^n C_1^n C_2^n} - \phi_{AC_1}^{\otimes n} \otimes \chi_{BC_2}^{\otimes n} \right\|_1 \leq \varepsilon_n \,, \tag{189}$$

by (183), where

$$\widehat{\sigma}_{A^n B^n C_1^n C_2^n} = (\mathbb{1}_{AB} \otimes V)^{\otimes n} \widehat{\rho}_{A^n B^n C^n} (\mathbb{1}_{AB} \otimes V^\dagger)^{\otimes n} \,. \tag{190}$$

It thus follows that

$$\left\| \widehat{\sigma}_{A^n C_1^n} - \phi_{AC_1}^{\otimes n} \right\|_1 \leq \varepsilon_n \tag{191}$$

and

$$\left\| \widehat{\sigma}_{B^n C_2^n} - \chi_{BC_2}^{\otimes n} \right\|_1 \leq \varepsilon_n \,. \tag{192}$$

Now, Alice's communication rate is bounded by

$$\begin{aligned}
2nQ_1 &\overset{(a)}{\geq} I(M_1; A^n | M_2)_{\rho^{(1)} \otimes \rho^{(2)}} \\
&\overset{(b)}{=} I(M_1 M_2; A^n)_{\rho^{(1)} \otimes \rho^{(2)}} \\
&\overset{(c)}{\geq} I(C^n; A^n)_{\widehat{\rho}} \\
&\overset{(d)}{=} I(C_1^n C_2^n; A^n)_{\widehat{\sigma}} \\
&\overset{(e)}{\geq} I(C_1^n C_2^n; A^n)_\omega - n\alpha_n
\end{aligned}$$

$$\overset{(f)}{=} 2nH(C_1)_\phi - n\alpha_n$$
$$\overset{(g)}{=} 2nH(A)_\omega - n\alpha_n \,, \tag{193}$$

where (a) holds because M_1 is of dimension 2^{nQ_1}, (b) since $I(M_2; A^n)_{\rho^{(1)}\otimes\rho^{(2)}} = 0$, (c) follows from the data processing inequality, (d) holds since the von Neumann entropy is isometrically invariant, (e) by the AFW inequality [46] , (f) since the mutual information is calculated with respect to the product state $|\phi_{AC_1}\rangle^{\otimes n} \otimes |\chi_{BC_2}\rangle^{\otimes n}$, and (g) holds since $|\phi_{AC_1}\rangle$ is a purification of ω_A. The bound on Bob's communication rate follows by symmetry. This completes the proof of Theorem 9. □

XII. Summary and discussion

We study coordination in three network models with classical communication links: 1) two-node network simulating a c-q state, 2) no-communication network simulating a separable state, and 3) a broadcast network simulating a c-q-q state, and consider coordination in additional three networks with quantum links: 1) a cascade network simulating quantum states with limited communication and entanglement assistance, 2) a quantum linked broadcast network simulating a joint quantum state with classical side information, and 3) a multiple-access network, generating entanglement between each sender and the receiver. We observe that the network topology dictates the type of states that can be simulated. Our findings generalize classical results from [26] and [40], and also quantum results from [8].

The results are relevant for various applications, where the network nodes could represent classical-quantum sensors [51], computers performing a joint computation task [52, 53], or players in a nonlocal game [54, 55] as we illustrated in the broadcast network with quantum links, in which we establishes the optimal rates required to achieve a certain quantum correlation to win a game at a desired probability. This work can be viewed as a step forward in understanding coordination in a general network that may comprise either classical or quantum resources. Another interesting direction for future work is the characterization of network coordination in the one-shot regime.

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