# Entanglement Coordination Rates in Multi-User Networks

Hosen Nator and Uzi Pereg

Electrical and Computer Engineering and Helen Diller Quantum Center, Technion

***Abstract*—The optimal coordination rates are determined in three primary settings of multi-user quantum networks, thus characterizing the minimal resources for simulating a joint quantum state among multiple parties. We study the following models: (1) a cascade network with limited entanglement, (2) a broadcast network, which consists of a single sender and two receivers, (3) a multiple-access network with two senders and a single receiver. We establish the necessary and sufficient conditions on the asymptotically-achievable communication and entanglement rates in each setting. At last, we show the implications of our results on nonlocal games with quantum strategies.**

## I. Introduction

State distribution and coordination are important in quantum communication [1], computation [2], and cryptography [3]. The quantum coordination problem can be described as follows. Consider a network that consists of $N$ nodes, where Node $i$ can perform an encoding operation $\mathcal{E}_i$ on a quantum system $A_i$, and its state should be in a certain correlation with the rest of the network nodes. The objective is to simulate a specific joint state $\omega_{A_1,A_2,\ldots,A_N}$. Node $i$ can send qubits to node $j$ via a quantum channel at a limited rate $Q_{i,j}$. The nodes may also share limited entanglement resources, prior to their communication. The optimal performance is characterized by the quantum communication rates $Q_{i,j}$ that are necessary and sufficient for simulating the desired quantum correlation. In another paper by the authors [4], we consider networks with classical communication links.

Instances of the network coordination problem include channel/source simulation [5–9], state merging [10, 11], state redistribution [12], entanglement dilution [13], randomness extraction [14], source coding [15, 16], and many other.

*Two-node classical coordination:* In classical coordination, the goal is to simulate a joint probability distribution. In the basic two-node network, as in Figure 1, a joint distribution $p_{XY}$ can be simulated if and only if the *classical* communication rate $R_{1,2}$ is above Wyner's common information, $C(X;Y) \triangleq \min I(U;XY)$, where the minimum is taken over all auxiliary variables $U$ that satisfy the Markov relation $X \multimap U \multimap Y$, and $I(U;XY)$ is the mutual information between $U$ and $(X,Y)$. One may also consider the case where the nodes share classical correlation resources, a priori, in the form of common randomness. Given a sufficient amount of pre-shared common randomness, the desired distribution can be simulated if and only if the classical communication rate is above the mutual information, i.e., $R_{1,2} \geq I(X;Y)$ [17, 18].

*Two-node quantum coordination:* In the quantum setting, the goal is to simulate a joint state. A bipartite state $\omega_{AB}$ can be simulated if and only if the quantum communication rate is above the von Neumann entropy [19], i.e., $Q_{1,2} \geq H(\omega_B)$. Now, suppose that the nodes share entanglement resources, prior to their communication, as illustrated in Figure 2. Given sufficient entanglement, the desired state can be simulated if and only if the quantum communication rate satisfies $Q_{1,2} \geq \frac{1}{2}I(A;B)_\omega$, where $I(A;B)_\omega$ is the quantum mutual information, by the quantum reverse Shannon theorem [20].

*Multi-node quantum coordination:* Here, we consider quantum coordination in three multi-party networks, motivated by applications such as the quantum Internet and quantum repeaters [21]. In each setting, we determine the optimal coordination rates, characterizing the minimal resources required in order to simulate a joint quantum state among multiple parties. First, we examine a cascade network that consists of three users, as depicted in Figure 3. Alice, Bob, and Charlie wish to simulate a joint quantum state $\omega_{ABC}$. Before communication begins, each party shares entanglement with their nearest neighbor, at a limited rate. Alice sends qubits to Bob at a rate $Q_{1,2}$, and thereafter, Bob sends qubits to Charlie at a rate $Q_{2,3}$. Next, we consider a broadcast network with one sender and two receivers, where the receivers are provided with classical sequences of information $X^n$ and $Y^n$. See Figure 4. In the third setting, we consider a multiple-access network, with two transmitters and a single receiver, as illustrated in Figure 5.

We further discuss the implications of our results on nonlocal quantum games. In particular, coordination in the broadcast network in Figure 4 can be viewed as a sequential game, where a coordinator (the sender) provides the players (the receivers) with quantum resources. In the course of the game, the referee sends questions, $X^n$ and $Y^n$, to each player, and they respond with $B^n$ and $C^n$. In order to win the game with a certain probability, the communication rates must satisfy the constraints with respect to an appropriate correlation.

In the analysis, we use different techniques for each setting. For the cascade network, we use state redistribution [12]. In the broadcast network, we assume that Alice does not have prior correlation with Bob and Charlie's resources $X^n$ and $Y^n$. Therefore, the of state redistribution [12] and side information [22] are not suitable. Instead, we use a quantum version of binning. In the analysis of the multiple-access network, we use the Schumacher compression protocol and the isometric relation that is dictated by the network topology.

The full version of this paper can be found in [23].

## II. Models and Results

We introduce three quantum coordination models and provide the required definitions. A quantum state is specified by a density operator, $\rho_A$, on the Hilbert space $\mathcal{H}_A$. Let $\Delta(\mathcal{H}_A)$ denote the set of all such density operators. Then, $A^n = A_1 \cdots A_n$ is a sequence such that $\rho_{A^n} \in \Delta(\mathcal{H}_A^{\otimes n})$. A quantum channel $\mathcal{E}_{A\to B}$ is a CPTP map.

### A. Cascade network

Consider the cascade network with rate-limited entanglement, as depicted in Figure 3. Alice can send qubits to Bob at a rate $Q_{1,2}$ and Bob can send qubits to Charlie at a rate $Q_{2,3}$. Moreover, Alice and Bob as well as Bob and Charlie share entanglement resources at rates $E_{1,2}$ and $E_{2,3}$ respectively.

Alice, Bob, and Charlie would like to simulate a joint state $\omega_{ABC}^{\otimes n}$, where $\omega_{ABC} \in \Delta(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$. Before communication begins, each party shares bipartite entanglement with their nearest neighbor, $\left|\Psi_{T_A T_B'}\right\rangle$ is shared between Alice and Bob, while $\left|\Theta_{T_B'' T_C}\right\rangle$ between Bob and Charlie. The coordination protocol begins with Alice preparing the state of her output $A^n$ and a "quantum description" $M_{1,2}$. She sends $M_{1,2}$ to Bob. As Bob receives $M_{1,2}$, he encodes the output $B^n$ and his own quantum description, $M_{2,3}$, and then sends $M_{2,3}$ to Charlie. Upon receiving $M_{2,3}$, Charlie encodes $C^n$. The transmissions $M_{1,2}$ and $M_{2,3}$ are limited to the quantum communication rates $Q_{1,2}$ and $Q_{2,3}$, while the pre-shared resources to the entanglement rates $E_{1,2}$ and $E_{2,3}$.

*Definition* 1. A $(2^{nQ_{1,2}}, 2^{nQ_{2,3}}, 2^{nE_{1,2}}, 2^{nE_{2,3}}, n)$ coordination code for the cascade network in Figure 3 consists of:

1) Two bipartite states $\left|\Psi_{T_A T_B'}\right\rangle$ and $\left|\Theta_{T_B'' T_C}\right\rangle$ on Hilbert spaces of dimension $2^{nE_{1,2}}$ and $2^{nE_{2,3}}$, respectively.
2) two Hilbert spaces, $\mathcal{H}_{M_{1,2}}$ and $\mathcal{H}_{M_{2,3}}$, of dimension $2^{nQ_{1,2}}$ and $2^{nQ_{2,3}}$, respectively, and
3) three encoding maps, $\mathcal{E}_{T_A\to A^n M_{1,2}}$, $\mathcal{F}_{M_{1,2}T_B' T_B''\to B^n M_{2,3}}$, $\mathcal{D}_{M_{2,3}T_C\to C^n}$, for Alice, Bob, and Charlie, respectively.

Alice applies the encoding map $\mathcal{E}_{T_A\to A^n M_{1,2}}$ on her share $T_A$ of the entanglement resources. This results in the output state $\rho^{(1)}_{A^n M_{1,2} T_B'} = \mathcal{E}_{T_A\to A^n M_{1,2}}(\Psi_{T_A T_B'})$. She sends $M_{1,2}$ to Bob. He uses $M_{1,2}$ along with his share $T_B' T_B''$ of the entanglement resources to encode by $\rho^{(2)}_{A^n B^n M_{2,3} T_C} = \mathcal{F}_{M_{1,2}T_B' T_B''\to B^n M_{2,3}}(\rho^{(1)}_{A^n M_{1,2} T_B'} \otimes \Theta_{T_B'' T_C})$. Bob sends $M_{2,3}$ to Charlie, whose encoding results in the final joint state, $\widehat{\rho}_{A^n B^n C^n} = \mathcal{D}_{M_{2,3}T_C\to C^n}(\rho^{(2)}_{A^n B^n M_{2,3} T_C})$. The objective is that $\widehat{\rho}_{A^n B^n C^n}$ is arbitrarily close to the desired state $\omega_{ABC}^{\otimes n}$.

*Definition* 2. A rate tuple $(Q_{1,2}, Q_{2,3}, E_{1,2}, E_{2,3})$ is achievable, if $\forall\ \varepsilon, \delta > 0$ and large $n$, there exists a $(2^{n(Q_{1,2}+\delta)}, 2^{n(Q_{2,3}+\delta)}, 2^{n(E_{1,2}+\delta)}, 2^{n(E_{2,3}+\delta)}, n)$ coordination code satisfying $\left\|\widehat{\rho}_{A^n B^n C^n} - \omega_{ABC}^{\otimes n}\right\|_1 \le \varepsilon$.

*Remark* 1. Coordination in the cascade network can also be represented as a resource inequality [24], $Q_{1,2}[q\to q]_{A\to B} + E_{1,2}[qq]_{AB} + Q_{2,3}[q\to q]_{B\to C} + E_{2,3}[qq]_{BC} \ge \langle \omega_{ABC} \rangle$.

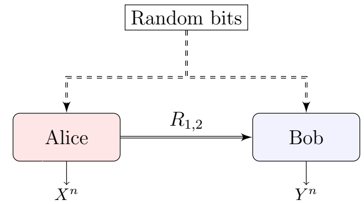

Fig. 1. Classical two-node network

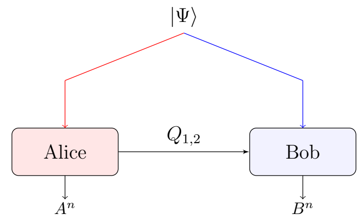

Fig. 2. Quantum two-node network

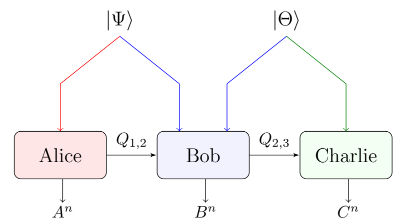

Fig. 3. Cascade network

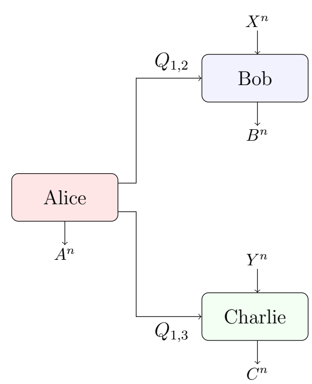

Fig. 4. Broadcast network

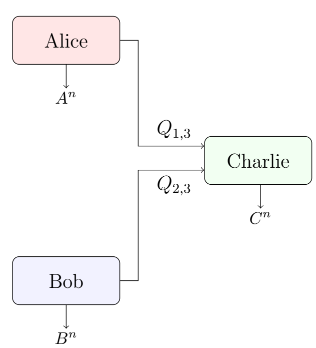

Fig. 5. Multiple access network

The optimal coordination rates are established below.

*Theorem* 1. Let $|\omega_{RABC}\rangle$ be a purification of $\omega_{ABC}$. A rate tuple $(Q_{1,2}, Q_{2,3}, E_{1,2}, E_{2,3})$ is achievable for coordination in the cascade network in Figure 3, if and only if

$$Q_{1,2} \geq \frac{1}{2} I(BC;R)_\omega\,, \tag{1}$$

$$Q_{1,2} + E_{1,2} \geq H(BC)_\omega\,, \tag{2}$$

$$Q_{2,3} \geq \frac{1}{2} I(C;RA)_\omega\,, \tag{3}$$

$$Q_{2,3} + E_{2,3} \geq H(C)_\omega\,. \tag{4}$$

The proof outline is provided in Section IV. The full proof can be found in [23].

### *B. Broadcast network*

Consider the broadcast network in Figure 4. Consider a classical-quantum state, $\omega_{XYABC}$, associated with a given ensemble of states $\left\{p_{XY}, \left|\sigma^{(x,y)}_{ABC}\right\rangle\right\}$ in $\Delta(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$. Alice, Bob, and Charlie wish to simulate $\omega_{XYABC}$. The classical sequences $X^n$ and $Y^n$ are drawn from a common source $p_{XY}^{\otimes n}$, and given to Bob and Charlie, respectively. Initially, Alice encodes her output $A^n$ with two quantum descriptions, $M_{1,2}$ and $M_{1,3}$, which she transmits to Bob and Charlie, respectively. Bob uses $M_{1,2}$ and $X^n$ to encode the output $B^n$. Similarly, Charlie receives $M_{1,3}$ and $Y^n$, and encodes his output $C^n$. Achievable rates are defined accordingly.

*Remark* 2. Broadcasting a quantum state among multiple receivers is impossible by the no-cloning theorem. However, in our broadcast network, Alice sends two different "quantum messages" $M_{1,2}$ and $M_{1,3}$, i.e., Alice broadcasts correlation.

*Remark* 3. Alice has no access to $X^n$ nor $Y^n$. Therefore, coordination can only be achieved for states $\omega_{XYABC}$ such that there is no correlation between $A$ and $XY$, on their own, i.e., $\omega_{XYA} = \omega_{XY} \otimes \omega_A$. Furthermore, standard techniques, such as state redistribution [12] and quantum source coding with side information [22], are not suitable. Instead, we introduce a quantum version of binning.

The optimal coordination rates are established below.

*Theorem* 2. A rate pair $(Q_{1,2}, Q_{1,3})$ for the broadcast network in Figure 4 is achievable if and only if

$$Q_{1,2} \geq H(B|X)_\omega\,,\ Q_{1,3} \geq H(C|Y)_\omega\,. \tag{5}$$

The proof outline for Theorem 2 is provided in Section V, and the full proof in [23].

### *C. Multiple access network*

Consider the multiple-access network in Figure 5. Alice, Bob, and Charlie would like to simulate a pure state $|\omega_{ABC}\rangle^{\otimes n}$. Alice can send qubits to Bob at a rate of $Q_{1,3}$ and Bob to Charlie at a rate of $Q_{2,3}$. A $(2^{nQ_{1,3}}, 2^{nQ_{2,3}}, n)$ code is defined accordingly. Alice and Bob apply the respective encoding map, $\mathcal{E}_{A^n \to A^n M_{1,3}}$ and $\mathcal{F}_{B^n \to B^n M_{2,3}}$, to prepare $\rho^{(1)}_{A^n M_{1,3}} \otimes \rho^{(2)}_{B^n M_{2,3}}$. As Charlie receives $M_{1,3}$ and $M_{2,3}$, and applies $\mathcal{D}_{M_{1,3}M_{2,3} \to C^n}$ to produce the final state, $\widehat{\rho}_{A^nB^nC^n}$. Achievable rates are defined accordingly.

*Remark* 4. As Charlie acts on $M_{1,3}$ and $M_{2,3}$, which are encoded separately, we have $\widehat{\rho}_{A^nB^n} = \rho^{(1)}_{A^n} \otimes \rho^{(2)}_{B^n}$. Therefore, one can only simulate states $\omega_{ABC}$ such that $\omega_{AB} = \omega_A \otimes \omega_B$. Hence, there exists an isometry $V_{C \to C_1C_2}$ such that

$$(\mathbb{1} \otimes V_{C \to C_1 C_2})\,|\omega_{ABC}\rangle = |\omega_{AC_1}\rangle \otimes |\omega_{BC_2}\rangle \tag{6}$$

where $|\omega_{AC_1}\rangle$ and $|\omega_{BC_2}\rangle$ are respective purifications [25, Theorem 5.1.1]. If $\omega_{ABC}$ cannot be decomposed as in (6), then coordination is impossible in this network.

The optimal coordination rates are given below.

*Theorem* 3. Let $|\omega_{ABC}\rangle$ be a pure state as in (6). Then, a rate pair $(Q_{1,3}, Q_{2,3})$ for coordination in the multiple-access network in Figure 5 is achievable if and only if

$$Q_{1,3} \geq H(A)_\omega\,,\ Q_{2,3} \geq H(B)_\omega\,. \tag{7}$$

The proof outline for Theorem 3 is provided in Section VI, and the full proof in [23].

## III. Nonlocal Games

The broadcast network model can represent a nonlocal game [26], where quantum coordination between the players could provide an advantage. First, we discuss the single-shot game, and then move on to sequential games with an asymptotic payoff. Consider Figure 4. Assume that $B$ and $C$ are classical, while $A$ is void. Here, Alice is a coordinator that generates correlation between the players, Bob and Charlie. In the sequential game, we denote the number of rounds by $n$.

*Single shot game ($n = 1$):* A referee provides two queries $X$ and $Y$, drawn at random, one for Bob and the other for Charlie. The players, provide responses, $B$ and $C$, respectively. They win the game if the tuple $(X, Y, B, C)$ satisfies a particular condition. A well known example is the CHSH game [27], where $X, Y, B, C \in \{0, 1\}$, and the winning condition is $X \wedge Y = B \oplus C$. Using classical correlations, the game can be won with probability (w.p.) of at most 0.75. If the players, Bob and Charlie, share a bipartite state $\rho_{M_{1,2}M_{1,3}}$, then they can generate a quantum correlation,

$$P_{BC|XY}(b, c|xy) = \operatorname{Tr}\left[\left(F^{(x)}_b \otimes D^{(y)}_c\right)\rho_{M_{1,2}M_{1,3}}\right] \tag{8}$$

by performing local measurements $\{F^{(x)}_b\}$ and $\{D^{(y)}_c\}$. In particular, in the CHSH game, if the coordinator, Alice, generates an EPR pair, $\left|\Phi_{M_{1,2}M_{1,3}}\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, then their chance of winning improves to $\cos^2\left(\frac{\pi}{8}\right) \approx 0.8535$. Here, $(Q_{1,2}, Q_{1,3}) = (1, 1)$ is optimal. In the Slofstra-Vidick game [28], winning w.p. $(1 - e^{-T})$ requires $Q_{1,j} \propto T$.

*Sequential game:* The players repeat the game $n$ times, and they can thus use a coordination code. Let $\mathscr{S}(\gamma)$ denote the set of correlations that win w.p. $\gamma$. Based on our results, the game can be won w.p. $\gamma$ if and only if Alice can send qubits to Bob and Charlie at rates $Q_{1,2}$ and $Q_{1,3}$ that satisfy the constraints in Theorem 2 for $P_{BC|XY} \in \mathscr{S}(\gamma)$.

## IV. Cascade Network Analysis

Consider the cascade network in Figure 3.

## A. Achievability proof

The proof for the direct part exploits state redistribution: Consider two parties, Alice and Bob. Let their systems be described by the joint state $\psi_{ABG}$, where $A$ and $B$ belong to Alice, and $G$ belongs to Bob. Let the state $|\psi_{ABGR}\rangle$ be a purification. They wish to redistribute so that $B$ is transferred from Alice to Bob. Alice can send quantum descriptions at rate $Q$ and they share entanglement at a rate $E$. Then, by the state redistribution theorem [12], the optimal rates satisfy

$$Q \geq \frac{1}{2} I(B;R|G)_\psi \,, \tag{9}$$

$$Q + E \geq H(B|G)_\psi \,. \tag{10}$$

We go back to the coordination setting for the cascade network (see Figure 3). Suppose that Alice prepares the desired state $|\omega_{A\bar{B}\bar{C}\bar{R}}\rangle^{\otimes n}$ locally, where $\bar{B}^n$, $\bar{C}^n$, $\bar{R}^n$ are her ancillas. Let $\varepsilon > 0$ be arbitrarily small. By the state redistribution theorem [12], Alice can transmit $\bar{B}^n\bar{C}^n$ to Bob at communication rate $Q_{1,2}$ and entanglement rate $E_{1,2}$ that satisfy (1)-(2). That is, there exist a bipartite state $\Psi_{T_A T_B'}$ and encoding maps, $\mathcal{E}^{(1)}_{\bar{B}^n\bar{C}^n T_A \to M_{1,2}}$ and $\mathcal{F}^{(1)}_{M_{1,2}T_B' \to B^n\widetilde{C}^n}$, such that

$$\left\| \tau^{(1)}_{\bar{R}^n A^n B^n \widetilde{C}^n} - \omega^{\otimes n}_{RABC} \right\|_1 \leq \varepsilon, \tag{11}$$

for sufficiently large $n$, where

$$\tau^{(1)}_{\bar{R}^n A^n B^n \widetilde{C}^n} = (\mathcal{F}^{(1)} \circ \mathcal{E}^{(1)}) \left( \omega^{\otimes n}_{RABC} \otimes \Psi_{T_A T_B'} \right) \,. \tag{12}$$

Similarly, $\bar{C}^n$ can be compressed and transmitted with rates as in (3)-(4). Namely, there exists $\Theta_{T_B'' T_C}$ and encoding maps, $\mathcal{F}^{(2)}_{\bar{C}^n T_B'' \to M_{2,3}}$ and $\mathcal{D}^{(2)}_{M_{2,3}T_C \to C^n}$, such that

$$\left\| \tau^{(2)}_{\bar{R}^n A^n B^n C^n} - \omega^{\otimes n}_{\bar{R}ABC} \right\|_1 \leq \varepsilon, \tag{13}$$

where $\tau^{(2)}_{\bar{R}^n A^n B^n C^n} = (\mathcal{D}^{(2)} \circ \mathcal{F}^{(2)}) \left( \omega^{\otimes n}_{\bar{R}A\bar{B}\bar{C}} \otimes \Theta_{T_B'' T_C} \right)$. The coding operations are described below.

*Encoding:*

A) Alice prepares $|\omega_{A\bar{B}\bar{C}\bar{R}}\rangle^{\otimes n}$ locally. She applies $\mathcal{E}^{(1)}$, and sends $M_{1,2}$ to Bob.
B) Bob receives $M_{1,2}$ and applies $\mathcal{F}^{(2)} \circ \mathcal{F}^{(1)}$.
C) Charlie receives $M_{2,3}$ from Bob and applies $\mathcal{D}^{(2)}$.

*Error analysis:* The joint state after Alice's encoding is $\rho^{(1)}_{A^n M_{1,2} T_B'} = \mathcal{E}^{(1)}(\omega^{\otimes n}_{A\bar{B}\bar{C}} \otimes \Psi_{T_A T_B'})$. After Bob encodes,

$$\rho^{(2)}_{A^n B^n M_{2,3} T_C} = \mathcal{F}^{(2)}(\tau^{(1)}_{A^n B^n \widetilde{C}^n} \otimes \Theta_{T_B'' T_C}) \tag{14}$$

(see (12)). According to (11), $\tau^{(1)}$ and $\omega^{\otimes n}$ are close in trace distance. By trace monotonicity under quantum channels,

$$\left\| \rho^{(2)}_{A^n B^n M_{2,3} T_C} - \mathcal{F}^{(2)}(\omega^{\otimes n}_{AB\widetilde{C}} \otimes \Theta_{T_B'' T_C}) \right\|_1 \leq \varepsilon. \tag{15}$$

As Charlie receives and encodes, the final state is $\widehat{\rho}_{A^n B^n C^n} = \mathcal{D}^{(2)}(\rho^{(2)}_{A^n B^n M_{2,3} T_C})$. Hence, by trace monotonicity,

$$\left\| \widehat{\rho}_{A^n B^n C^n} - \tau^{(2)}_{A^n B^n C^n} \right\|_1 \leq \varepsilon \,. \tag{16}$$

(see (13)). Thus, by (13), (16), and the triangle inequality, $\left\| \widehat{\rho}_{A^n B^n C^n} - \omega^{\otimes n}_{ABC} \right\|_1 \leq 2\varepsilon$. This proves achievability.

## B. Converse proof

Let $(Q_{1,2}, Q_{2,3}, E_{1,2}, E_{2,3})$ be an achievable rate tuple. Suppose that Alice prepares the state $|\omega_{R\bar{A}\bar{B}\bar{C}}\rangle^{\otimes n}$ locally, and then performs the coordination protocol. Then, there exists a sequence of codes such that

$$\left\| \widehat{\rho}_{R^n A^n B^n C^n} - \omega^{\otimes n}_{RABC} \right\|_1 \leq \varepsilon_n \,. \tag{17}$$

Consider Alice's communication and entanglement rates, $Q_{1,2}$ and $E_{1,2}$. Now,

$$2n(Q_{1,2} + E_{1,2}) \geq I(M_{1,2}T_B'; A^n R^n)_{\rho^{(1)}} \tag{18}$$

since $I(A;B)_\rho \leq 2 \log \dim(\mathcal{H}_A)$ in general. We may view the entire encoding operation of Bob and Charlie as a black box whose input and output are $(M_{1,2}, T_B')$ and $(B^n, C^n)$, respectively. Then, by the data processing inequality,

$$\begin{aligned} I(M_{1,2}T_B'; A^n R^n)_{\rho^{(1)}} &\geq I(B^n C^n; A^n R^n)_{\widehat{\rho}} \\ &\geq I(B^n C^n; A^n R^n)_{\omega^{\otimes n}} - n\alpha_n \\ &= n[I(BC;AR)_\omega - \alpha_n] \,, \end{aligned} \tag{19}$$

where $\alpha_n \to 0$ when $n \to \infty$. The second inequality follows from (17) and the AFW inequality [29]. Since $|\omega_{RABC}\rangle$ is pure, we have $I(BC;AR)_\omega = 2H(BC)_\omega$. Therefore, $Q_{1,2} + E_{1,2} \geq H(BC)_\omega - \frac{1}{2}\alpha_n$.

To show the bound on $Q_{1,2}$, observe that a lower bound on the communication rate with unlimited entanglement also holds with limited resources. Thus, $Q_{1,2} \geq \frac{1}{2} I(BC;R)_\omega$ follows from the entanglement-assisted capacity theorem [17]. The bound on Bob's rates follows in a similar manner. ☐

## V. Broadcast Analysis

Consider the broadcast network in Figure 4. We show achievability by using a quantum version of the binning technique. Define the average states,

$$\sigma^{(x)}_{AB} = \sum_{y \in \mathcal{Y}} p_{Y|X}(y|x) \sigma^{(x,y)}_{AB} \,,\; \sigma^{(y)}_{AC} = \sum_{x \in \mathcal{X}} p_{X|Y}(x|y) \sigma^{(x,y)}_{AC} \,,$$

and consider a spectral decomposition of the reduced states,

$$\sigma^{(x)}_B = \sum_{z \in \mathcal{Z}} p_{Z|X}(z|x) \, |\psi_{x,z}\rangle\!\langle\psi_{x,z}| \,, \tag{20}$$

$$\sigma^{(y)}_C = \sum_{w \in \mathcal{W}} p_{W|Y}(w|y) \, |\phi_{y,w}\rangle\!\langle\phi_{y,w}| \,. \tag{21}$$

We can also assume that the different bases are orthogonal to each other by requiring that Bob and Charlie encode on a different Hilbert space for every value of $(x,y)$.

We use the type class definitions and notations in [25, Chap. 14]. In particular, $T_\delta^{X^n}$ denotes the $\delta$-typical set with respect to $p_X$, and $T_\delta^{Z^n|x^n}$ is the conditional $\delta$-typical set with respect to $p_{XZ}$, given $x^n \in T_\delta^{X^n}$.

*Classical Codebook Generation:* For every sequence $z^n \in \mathcal{Z}^n$, assign an index $m_{1,2}(z^n)$, uniformly from $[2^{nQ_{1,2}}]$. A bin $\mathfrak{B}_{1,2}(m_{1,2})$ is defined as the subset of sequences that are assigned the index $m_{1,2}$. Define the bins $\mathfrak{B}_{1,3}(m_{1,3})$ similarly.

*Encoding:*

A) Alice prepares $\omega^{\otimes n}_{A\bar{B}\bar{C}}$ locally, without any correlation with $X^n$ and $Y^n$ (see Remark 3). She applies the encoding channel $\mathcal{E}^{(1)}_{\bar{B}^n\to M_{1,2}} \otimes \mathcal{E}^{(2)}_{\bar{C}^n\to M_{1,3}}$, where

$$\mathcal{E}^{(1)}(\rho) = \sum_{x^n\in\mathcal{X}^n} p_X^{\otimes n}(x^n) \sum_{z^n\in\mathcal{Z}^n} \langle\psi_{x^n,z^n}|\,\rho_1\,|\psi_{x^n,z^n}\rangle\,|m_{1,2}(z^n)\rangle\!\langle m_{1,2}(z^n)|\;, \qquad (22)$$

and $\mathcal{E}^{(2)}$ is define similarly. She transmits $M_{1,2}$ and $M_{1,3}$.

B) Bob applies the following encoding channel,

$$\mathcal{F}^{(x^n)}_{M_{1,2}\to B^n}(\rho_{M_{1,2}}) = \sum_{m_{1,2}=1}^{2^{nQ_1}} \langle m_{1,2}|\,\rho_{M_{1,2}}\,|m_{1,2}\rangle \cdot \frac{1}{\left|T_\delta^{Z^n|x^n}\cap\mathfrak{B}_{1,2}(m_{1,2})\right|} \sum_{\substack{z^n\in T_\delta^{Z^n|x^n}\\ \cap\mathfrak{B}_{1,2}(m_{1,2})}} |\psi_{x^n,z^n}\rangle\!\langle\psi_{x^n,z^n}| \qquad (23)$$

C) Charlie's decoder is defined in a similar manner.

*Error analysis:* We focus on Bob's error. Consider a given codebook $\mathscr{C}_{1,2} = \{m_{1,2}(z^n)\}$. Alice encodes $M_{1,2}$ by

$$\mathcal{E}^{(1)}(\omega^{\otimes n}_{AB}) = \sum_{\tilde{x}^n\in\mathcal{X}^n} p_X^{\otimes n}(\tilde{x}^n) \sum_{z^n\in\mathcal{Z}^n} \langle\psi_{\tilde{x}^n,z^n}|\,\omega^{\otimes n}_{AB}\,|\psi_{\tilde{x}^n,z^n}\rangle\,|m_{1,2}(z^n)\rangle\!\langle m_{1,2}(z^n)|\;, \qquad (24)$$

where $|\psi\rangle_{x^n,z^n} \equiv \bigotimes_{i=1}^n |\psi\rangle_{x_i,z_i}$. By the weak law of large numbers, this state is $\varepsilon_1$-close in trace distance to $\rho^{(1)}_{A^nM_{1,2}} = \sum_{x^n\in T_\delta^{X^n}} p_X^{\otimes n}(x^n)\rho^{(1|x^n)}_{A^nM_{1,2}}$ where we have defined

$$\rho^{(1|x^n)}_{A^nM_{1,2}} = \sum_{z^n\in T_\delta^{Z^n|x^n}} \langle\psi_{x^n,z^n}|\,\sigma^{(x^n)}_{A^n\bar{B}^n}\,|\psi_{x^n,z^n}\rangle\,|m_{1,2}(z^n)\rangle\!\langle m_{1,2}(z^n)|\;. \qquad (25)$$

Let $x^n\in T_\delta^{X^n}$. By the definition of Bob's encoding channel,

$$\mathcal{F}^{(x^n)}(\rho^{(1|x^n)}_{A^nM_{1,2}}) = \sum_{z^n\in T_\delta^{Z^n|x^n}} \langle\psi_{x^n,z^n}|\,\sigma^{(x^n)}_{A^n\bar{B}^n}\,|\psi_{x^n,z^n}\rangle \otimes \frac{1}{\left|T_\delta^{Z^n|x^n}\cap\mathfrak{B}_{1,2}(m_{1,2}(z^n))\right|} \sum_{\substack{\tilde{z}^n\in T_\delta^{Z^n|x^n}\\ \cap\mathfrak{B}_{1,2}(m_{1,2}(z^n))}} |\psi_{x^n,\tilde{z}^n}\rangle\!\langle\psi_{x^n,\tilde{z}^n}|\,. \qquad (26)$$

Based on the classical result [30, Chapter 10.3], the random codebook $\mathscr{C}_{1,2}$ satisfies that

$$\Pr_{\mathscr{C}_{1,2}}\left(\exists\tilde{z}^n\in T_\delta^{Z^n|x^n}\cap\mathfrak{B}_{1,2}(m_{1,2}(z^n)) : \tilde{z}^n\neq z^n\right) \leq \varepsilon_2$$

given $z^n \in T_\delta^{Z^n|x^n}$, for sufficiently large $n$, provided that the codebook size is at least $2^{n(H(Z|X)+\varepsilon_3)}$, i.e., if $Q_{1,2} > H(Z|X)+\varepsilon_3 = H(B|X)_\omega + \varepsilon_3$. Observe that if $T_\delta^{Z^n|x^n}\cap\mathfrak{B}_{1,2}(m_{1,2}(z^n))$ consists of the sequence $z^n$ alone, then the overall state in (26) is identical to the post-measurement state after a typical subspace measurement on $B^n$, with respect to $T_\delta^{Z^n|x^n}$. Based on the gentle measurement lemma [31], this state is $\varepsilon_4$-close to $\sigma^{(x^n)}_{AB}$, for sufficiently large $n$.

Using the triangle inequality and the total expectation formula, we show that $\mathbb{E}_{\mathscr{C}_{1,2}}\left\|\omega^{\otimes n}_{XAB} - \widehat{\rho}_{X^nA^nB^n}\right\| \leq \varepsilon_1+\varepsilon_2+\varepsilon_4$.

By symmetry, Charlie's error tends to zero as well, provided that $Q_{1,3} \geq H(C|Y)_\omega + \varepsilon_5$. The achievability proof follows by taking $n\to\infty$ and then $\varepsilon_j,\delta\to 0$. The converse proof follows the lines of [12], and it is thus omitted. □

## VI. Multiple-Access Analysis

Consider the multiple-access network in Figure 5. Coordination is only possible if (6) holds for some isometry $V$ (see Remark 4). Thus, assume so. Achievability follows from the Schumacher compression protocol [32] straightforwardly. Alice and Bob prepare $\omega^{\otimes n}_{AC_1}\otimes\omega^{\otimes n}_{BC_2}$, compress and send $C_1^n$ and $C_2^n$, and Charlie applies $(V^\dagger)^{\otimes n}$. The details are omitted.

As for the converse, consider a sequence of codes such that $\left\|\widehat{\rho}_{A^nB^nC^n} - \omega^{\otimes n}_{ABC}\right\|_1 \leq \varepsilon_n$ vanish. Applying $V^{\otimes n}$ yields

$$\left\|\widehat{\sigma}_{A^nB^nC_1^nC_2^n} - \omega^{\otimes n}_{AC_1}\otimes\omega^{\otimes n}_{BC_2}\right\|_1 \leq \varepsilon_n\,. \qquad (27)$$

Thus,

$$\begin{aligned} 2nQ_{1,3} &\geq I(M_{1,3};A^n|M_{2,3})_{\rho^{(1)}\otimes\rho^{(2)}} \\ &\overset{(a)}{=} I(M_{1,3}M_{2,3};A^n)_{\rho^{(1)}\otimes\rho^{(2)}} \\ &\overset{(b)}{\geq} I(C^n;A^n)_{\widehat{\rho}} \overset{(c)}{=} I(C_1^nC_2^n;A^n)_{\widehat{\sigma}} \\ &\overset{(d)}{\geq} I(C_1^nC_2^n;A^n)_\omega - n\alpha_n \\ &\overset{(e)}{=} 2nH(A)_\omega - n\alpha_n\,, \end{aligned} \qquad (28)$$

since $(a)$ $I(M_{2,3};A^n)_{\rho^{(1)}\otimes\rho^{(2)}} = 0$, $(b)$ the data processing inequality, $(c)$ the von Neumann entropy is isometrically invariant, $(d)$ the AFW inequality [29], and $(e)$ the mutual information is with respect to $|\omega_{AC_1}\rangle^{\otimes n}\otimes|\omega_{BC_2}\rangle^{\otimes n}$, and therefore $H(A)_\omega = H(C_1)_\omega$. The bound on Bob's rate follows by symmetry. Further details are given in [23]. □

## VII. Summary

We considered quantum coordination in three models. In the cascade network, three users simulate a joint state using limited communication and entanglement rates. Next, we considered a broadcast network, where a sender broadcasts correlation to two receivers. We discussed the implications of our results on nonlocal games. At last, we considered a multiple-access network with two senders and a single receiver. We observed that the network topology dictates the type of states that can be simulated. Our results extend various results in the literature and can be generalized to more than three nodes.

## Acknowledgments

H. Nator and U. Pereg were supported by ISF Grants n. 939/23 and 2691/23, DIP n. 2032991, and Nevet Program of the Helen Diller Quantum Center at the Technion, n. 2033613. U. Pereg was also supported by the Israel VATAT Program for Quantum Science and Technology n. 86636903, and the Chaya Career Advancement Chair, n. 8776026.

## REFERENCES


[1] M. Hsieh and M. M. Wilde, "Trading classical communication, quantum communication, and entanglement in quantum shannon theory," *IEEE Trans. Inf. Theory*, vol. 56, no. 9, pp. 4705–4730, Sep 2010.

[2] J. M. Matera, D. Egloff, N. Killoran, and M. B. Plenio, "Coherent control of quantum systems as a resource theory," *Quantum Sci. Tech.*, vol. 1, no. 1, p. 01LT01, 2016.

[3] Y. Liu, T.-Y. Chen, L.-J. Wang, H. Liang, G.-L. Shentu, J. Wang, K. Cui, H.-L. Yin, N.-L. Liu, L. Li *et al.*, "Experimental measurement-device-independent quantum key distribution," *Phys. Rev. Lett.*, vol. 111, no. 13, p. 130502, 2013.

[4] H. Nator and U. Pereg, "Coordination capacity for classical-quantum correlations," *submitted to ITW 2024. Preprint available in* `arXiv:2404.18297` **[quant-ph]**, 2024.

[5] M. X. Cao, R. Jain, and M. Tomamichel, "Quantum channel simulation under purified distance is no more difficult than state splitting," *arXiv preprint arXiv:2403.14416*, 2024.

[6] M. Berta, F. G. Brandão, M. Christandl, and S. Wehner, "Entanglement cost of quantum channels," *IEEE Trans. Inf. Theory*, vol. 59, no. 10, pp. 6779–6795, 2013.

[7] M. M. Wilde, "Entanglement cost and quantum channel simulation," *Phys. Rev. A*, vol. 98, no. 4, p. 042338, 2018.

[8] K. Li and Y. Yao, "Quantum broadcast channel simulation via multipartite convex splitting," *arXiv preprint arXiv:2112.04475*, 2021.

[9] I. George, M.-H. Hsieh, and E. Chitambar, "One-shot bounds on state generation using correlated resources and local encoders," in *IEEE Int. Symp. Inf. Theory (ISIT'2023)*, 2023, pp. 96–101.

[10] I. Bjelaković, H. Boche, and G. Janßen, "Universal quantum state merging," *J. Math. Phys.*, vol. 54, no. 3, 2013.

[11] M. Horodecki, J. Oppenheim, and A. Winter, "Quantum state merging and negative information," *Commun. Math. Phys.*, vol. 269, pp. 107–136, 2007.

[12] J. T. Yard and I. Devetak, "Optimal quantum source coding with quantum side information at the encoder and decoder," *IEEE Trans. Inf. Theory*, vol. 55, no. 11, pp. 5339–5351, 2009.

[13] F. Salek and A. Winter, "Multi-user distillation of common randomness and entanglement from quantum states," *IEEE Trans. Inf. Theory*, vol. 68, no. 2, pp. 976–988, 2022.

[14] M. Tahmasbi and M. R. Bloch, "Steganography protocols for quantum channels," *J. Math. Phys.*, vol. 61, no. 8, 2020.

[15] E. Soljanin, "Compressing quantum mixed-state sources by sending classical information," *IEEE Trans. Inf. Theory*, vol. 48, no. 8, pp. 2263–2275, 2002.

[16] H. M. Garmaroudi, S. S. Pradhan, and J. Chen, "Rate-limited quantum-to-classical optimal transport: A lossy source coding perspective," in *IEEE Int. Symp. Inf. Theory (ISIT'2023)*, 2023, pp. 1925–1930.

[17] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V. Thapliyal, "Entanglement-assisted capacity of a quantum channel and the reverse Shannon theorem," *IEEE Trans. Inf. Theory*, vol. 48, no. 10, pp. 2637–2655, Oct 2002.

[18] P. Cuff, "Communication requirements for generating correlated random variables," in *2008 IEEE Int. Symp. Inf. Theory*, 2008, pp. 1393–1397.

[19] A. W. Harrow and H.-K. Lo, "A tight lower bound on the classical communication cost of entanglement dilution," *IEEE Trans. Inf. Theory*, vol. 50, no. 2, pp. 319–327, 2004.

[20] C. H. Bennett, I. Devetak, A. W. Harrow, P. W. Shor, and A. Winter, "The quantum reverse Shannon theorem and resource tradeoffs for simulating quantum channels," *IEEE Trans. Inf. Theory*, vol. 60, no. 5, pp. 2926–2959, 2014.

[21] M. Razavi, "Multiple-access quantum key distribution networks," *IEEE Trans. Commun.*, vol. 60, no. 10, pp. 3071–3079, 2012.

[22] Z. Baghali Khanian and A. Winter, "Distributed compression of correlated classical-quantum sources or: The price of ignorance," *IEEE Trans. Inf. Theory*, vol. 66, no. 9, pp. 5620–5633, 2020.

[23] H. Nator and U. Pereg, "Entanglement coordination rates in multi-user networks," *submitted to ITW 2024. Preprint available in* `arXiv:2403.11893` **[quant-ph]**, 2024. [Online]. Available: https://arxiv.org/abs/2403.11893

[24] I. Devetak, A. W. Harrow, and A. J. Winter, "A resource framework for quantum shannon theory," *IEEE Trans. Inf. Theory*, vol. 54, no. 10, pp. 4587–4618, Oct 2008.

[25] M. M. Wilde, *Quantum Information Theory*, 2nd ed. Cambridge University Press, 2017.

[26] R. Cleve, P. Hoyer, B. Toner, and J. Watrous, "Consequences and limits of nonlocal strategies," in *Proc. 19th IEEE Annu. Conf. Comput. Complexity, 2004.* IEEE, 2004, pp. 236–249.

[27] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, "Proposed experiment to test local hidden-variable theories," *Phys. rev. lett.*, vol. 23, no. 15, p. 880, 1969.

[28] W. Slofstra and T. Vidick, "Entanglement in non-local games and the hyperlinear profile of groups," *Annales Henri Poincaré*, vol. 19, no. 10, pp. 2979–3005, 2018.

[29] A. Winter, "Tight uniform continuity bounds for quantum entropies: conditional entropy, relative entropy distance and energy constraints," *Commun. Math. Phys.*, vol. 347, pp. 291–313, 2016.

[30] A. E. Gamal and Y.-H. Kim, *Network Information Theory*. Cambridge University Press, 2011.

[31] A. Winter, "Coding theorem and strong converse for quantum channels," *IEEE Trans. Inf. Theory*, vol. 45, no. 7, pp. 2481–2485, 1999.

[32] B. Schumacher, "Quantum coding," *Phys. Rev. A*, vol. 51, no. 4, p. 2738, 1995.