Coordination Capacity for Classical-Quantum States

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***Abstract*—Network coordination is considered in three basic settings, characterizing the generation of separable and classical-quantum correlations among multiple parties. First, we consider the simulation of a classical-quantum state between two nodes with rate-limited common randomness (CR) and communication. Furthermore, we study the preparation of a separable state between multiple nodes with rate-limited CR and no communication. At last, we consider a broadcast setting, where a sender and two receivers simulate a classical-quantum-quantum state using rate-limited CR and communication. We establish the optimal tradeoff between communication and CR rates in each setting.**

***Index Terms*—Quantum communication, coordination, reverse Shannon theorem.**

I. Introduction

Coordination is essential in communication systems, as it ensures that different components can work together in harmony to achieve a common goal. For example, in sensor networks, the sensors do not just share information in the conventional sense, but also collaborate in the transmission of data [1]. Furthermore, coordination plays a major role in cooperative computing between distributed systems [2], rate-distortion theory for secrecy systems [3], simulation of distributed quantum measurements [4], and quantum non-local games [5]. It is envisioned that quantum information technology will enhance future communication systems from different perspectives, such as efficiency [6], security [7], and computing [8].

These advances motivate the study of coordination in quantum networks.

Cuff et al. [9] studied classical coordination in various communication networks with different topologies, and considered two coordination types, empirical coordination and strong coordination. Empirical coordination requires that the average frequency of joint actions in the network approaches a desired distribution with high certainty. On the other hand, strong coordination sets a requirement on the joint distribution of all actions. Efficient coordination codes are constructed in [10]. Source coding [11–14], state coordination [15], channel simulation [16–21], and distributed source simulation [22, 23] can be viewed as instances of network coordination. In particular, Bennet et al. [18] considered simulation of a quantum channel, under the assumption that pre-shared entanglement is available to the sender and the receiver, and derived the quantum reverse Shannon theorem [24, 25]. The optimal simulation rate turns out to be identical to the entanglement-assisted quantum capacity [18].

The general problem of quantum coordination can be formulated as follows. Consider a quantum network that consists of m nodes, where Node i performs an encoding operation $\mathcal{E}_i$ on a quantum system A_i, which is required to be in a certain desired correlation with the rest of the network. In other words, the goal is to simulate a particular joint state, $\omega_{A_1 A_2 \cdots A_m}$. In general, some of the nodes are not free to choose their encoding, but rather their state is dictated by Nature, according to a certain physical process. Node i can also send a sequence of bits or qubits to Node j via a communication link of a limited rate, $R_{i,j}$. The ultimate performance is defined by the set of rates $\{R_{i,j}\}$ that are necessary and sufficient in order to simulate the quantum correlation.

Cuff et al. [9] introduced the classical version of this problem, where the encoders, decoders, and rates are all classical, and the goal is to simulate a prescribed probability distribution. In the basic two-node setting, the simulation of a joint distribution p_{XY} requires a rate $R_{1,2} \geq C(X;Y)$, where $C(X;Y)$ is Wyner's common information [26]. The quantum analog was recently established by George et al. [23], in the context of distributed source simulation. Under the assumption that Alice and Bob share unlimited common randomness (CR) a priori, simulation can be performed at a lower rate, $R_{1,2} \geq I(X;Y)$, where $I(X;Y)$ is the mutual information. The capacity region describes the optimal tradeoff between the communication and CR rates [27].

In this paper, we consider three coordination settings. First, we consider the simulation of a classical-quantum state ω_{XB} between two nodes with rate-limited CR. We characterize the optimal tradeoff between the required rate of description and the amount of CR used. Our second model is a quantum no-communication network. The network comprises three nodes, where no-communication is allowed between the nodes, yet CR is available at a classical rate R_0. Thereby, only separable states can be simulated. We show that a joint state ω_{ABC} can be simulated at a CR rate of $R_0 \geq I(ABC;U)$, where U is an auxiliary classical random variable that satisfies a Markov property. At last, we consider a broadcast setting, where a sender and two receivers simulate a classical-quantum-quantum (c-q-q) state using rate-limited CR and communication. We establish the optimal tradeoff between communication and CR rate. In the analysis, we use random coding and apply quantum resolvability results [28, 29].

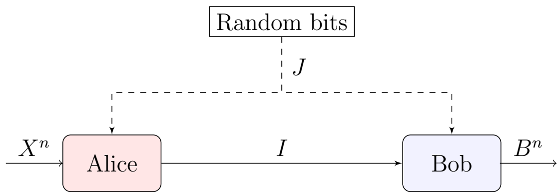

Fig. 1. Two-node network.

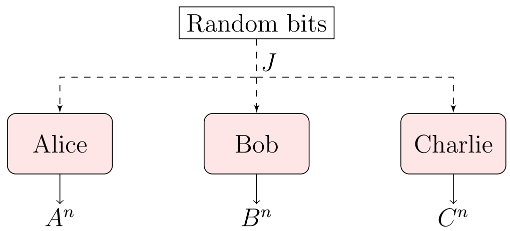

Fig. 2. No-communication network

II. Problem Definitions

We consider three coordination settings as described below. We use standard notation in quantum information theory, as in [30], $X, Y, Z, \ldots$ are discrete random variables on finite alphabets $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots$, respectively, $x^n = (x_i)_{i\in[n]}$ denotes a sequence in $\mathcal{X}^n$. A quantum state is described by a density operator, ρ_A, on the Hilbert space $\mathcal{H}_A$. Denote the set of all such operators by $\mathfrak{S}(\mathcal{H}_A)$. A c-q channel is a map $\mathcal{N}_{X\to B} : \mathcal{X} \to \mathfrak{S}(\mathcal{H}_B)$. The quantum mutual information is defined as $I(A;B)_\rho = H(\rho_A) + H(\rho_B) - H(\rho_{AB})$, where $H(\rho) \equiv -\mathrm{Tr}[\rho \log(\rho)]$, the conditional quantum entropy as $H(A|B)_\rho = H(\rho_{AB}) - H(\rho_B)$, and $I(A;B|C)_\rho$, accordingly.

A. Two-Node Network

Consider the two-node network in Figure 1. Alice and Bob wish to simulate a c-q state $\omega_{XB}^{\otimes n}$, using the following scheme. Node 1 (Alice) receives a classical source sequence x^n, drawn by Nature according to a given PMF p_X. The source sequence is encoded into an index i at a rate R_1. Node 2 (Bob) is quantum. Both nodes have access to a CR element j at a given rate R_0, i.e., j is uniformly distributed over $\left[2^{nR_0}\right]$, and it is independent of X^n.

Formally, a $\left(2^{nR_0}, 2^{nR_1}, n\right)$ coordination code for the simulation of a c-q state ω_{XB} consists of a classical encoding channel, $F : \mathcal{X}^n \times [2^{nR_0}] \to [2^{nR_1}]$, and a c-q decoding channel $\mathcal{D}_{IJ\to B^n}$. The protocol works as follows. A classical sequence $x^n \sim p_X^n$ is generated by Nature. Given the sequence x^n and the CR element j, Alice selects a random index,

$$i \sim F(\cdot|x^n, j) \tag{1}$$

and sends it through a noiseless link. As Bob receives the message i and the CR element j, he prepares the state

$$\rho_{B^n}^{(i,j)} = \mathcal{D}_{IJ\to B^n}(i,j)\,. \tag{2}$$

Hence, the resulting joint state is

$$\widehat{\rho}_{X^nB^n} = \frac{1}{2^{nR_0}} \sum_{j\in[2^{nR_0}]} \sum_{x^n\in\mathcal{X}^n} \Bigg(p_X^n(x^n)\, |x^n\rangle\!\langle x^n|_{X^n} \otimes \sum_{i\in[2^{nR_1}]} F(i|x^n,j)\rho_{B^n}^{(i,j)} \Bigg). \tag{3}$$

Definition 1. A coordination rate pair (R_0, R_1) is achievable for the simulation of ω_{XB}, if for every $\varepsilon > 0$ and sufficiently large n, there exists a $\left(2^{nR_0}, 2^{nR_1}, n\right)$ code that achieves

$$\left\|\widehat{\rho}_{X^nB^n} - \omega_{XB}^{\otimes n}\right\|_1 \le \varepsilon\,. \tag{4}$$

The coordination capacity region of the two-node network, $\mathcal{R}_{\text{2-node}}(\omega)$, with respect to the c-q state ω_{XB}, is the closure of the set of all achievable rate pairs.

The coordination capacity, $\mathsf{C}_{\text{2-node}}^{(0)}(\omega)$, without CR, is the supremum of rates R_1 such that $(0, R_1) \in \mathcal{R}_{\text{2-node}}(\omega)$. The CR-assisted coordination capacity, $\mathsf{C}_{\text{2-node}}^{(\infty)}(\omega)$, i.e., with unlimited CR, is the supremum of rates R_1 such that $(R_0, R_1) \in \mathcal{R}_{\text{2-node}}(\omega)$ for some $R_1 \ge 0$.

B. No-Communication Network

Consider a network that consists of three users: Alice, Bob and Charlie, holding quantum systems A, B, and C, respectively. The users cannot communicate, but they share a CR element j at a rate R_0, as illustrated in Figure 2. Given j, each user prepares a quantum state separately.

A $\left(2^{nR_0}, n\right)$ coordination code for the no-communication network consists of a CR set $[2^{nR_0}]$, and three c-q encoding channels, $\mathcal{T}_{J\to A^n}^{(1)}$, $\mathcal{T}_{J\to B^n}^{(2)}$, and $\mathcal{T}_{J\to C^n}^{(3)}$. As Alice, Bob, and Charlie receive a realization j of the CR element, each uses their encoding map to prepare their respective state. prepares a quantum state, $\rho_{A^n}^j = \mathcal{T}_{J\to A^n}^{(1)}(j)$, $\rho_{B^n}^j = \mathcal{T}_{J\to A^n}^{(2)}(j)$, and $\rho_{C^n}^j = \mathcal{T}_{J\to C^n}^{(3)}(j)$, respectively. Hence,

$$\widehat{\rho}_{A^nB^nC^n} = \frac{1}{2^{nR_0}} \sum_{j\in[2^{nR_0}]} \mathcal{T}^{(1)}(j) \otimes \mathcal{T}^{(2)}(j) \otimes \mathcal{T}^{(3)}(j)\,.$$

Definition 2. A CR rate R_0 is achievable for the simulation of ω_{ABC}, if for every $\varepsilon > 0$ and sufficiently large n, there exists a $\left(2^{nR_0}, n\right)$ coordination code that achieves

$$\left\|\widehat{\rho}_{A^nB^nC^n} - \omega_{ABC}^{\otimes n}\right\|_1 \le \varepsilon\,. \tag{5}$$

The coordination capacity $\mathsf{C}_{\text{NC}}(\omega)$, for the no-communication network, is the infimum of achievable rates R_0. If there are no achievable rates, we set $\mathsf{C}_{\text{NC}}(\omega) = +\infty$.

C. Broadcast Network

Consider the broadcast network in Figure 3. A sender, Alice, and two receivers, Bob 1 and Bob 2, wish to simulate a c-q-q state $\omega_{XB_1B_2}$, using the following scheme. Alice receives a classical source sequence $x^n \in \mathcal{X}^n$ drawn by Nature, i.i.d. according to a given PMF p_X. Alice encodes the source sequence into an index i at a rate R_1. The other two nodes, of Bob 1 and Bob 2, are quantum. The three nodes have access to a CR element j at a rate R_0. Similarly, a $\left(2^{nR_0}, 2^{nR_1}, n\right)$

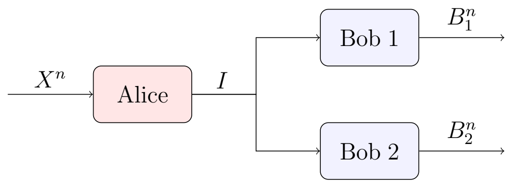

Fig. 3. Broadcast Network. The CR element is omitted for simplicity.

coordination code consists of a classical encoding channel, $F : \mathcal{X}^n \times [2^{nR_0}] \to [2^{nR_1}]$, and two c-q decoding channels, $\mathcal{D}^{(\ell)}_{IJ\to B^n_\ell}$, for $\ell \in \{1, 2\}$. Given x^n and the CR element j, Alice generates $i \sim F(\cdot|x^n, j)$, and sends it to both Bob 1 and Bob 2, who then apply their decoding map.

The coordination capacity region of the broadcast network, $\mathcal{R}_{\text{BC}}(\omega)$, with respect to the c-q state $\omega_{XB_1B_2}$, is defined in a similar manner as in Definition 1.

III. Results

A. Two-Nodes Network

Consider a given c-q state ω_{XB} that we wish to simulate. We now state our main result. Define the following set of c-q states. Let $\mathscr{S}_{\text{2-node}}(\omega)$ be the set of all c-q states

$$\sigma_{XUB} = \sum_{\substack{(x,u)\in \\ \mathcal{X}\times\mathcal{U}}} p_{X,U}(x,u)\, |x\rangle\langle x|_X \otimes |u\rangle\langle u|_U \otimes \theta^u_B \tag{6a}$$

such that

$$\sigma_{XB} = \omega_{XB} \tag{6b}$$

for $|\mathcal{U}| \leq |\mathcal{X}|^2[\dim(\mathcal{H}_B)]^2 + 1$. Notice that given a classical value $U = u$, there is no correlation between X and B.

Theorem 1. The coordination capacity region for the two-node system described in Figure 1 is the set

$$\mathcal{R}_{\text{2-node}}(\omega) = \bigcup_{\mathscr{S}_{\text{2-node}}(\omega)} \left\{ \begin{array}{l} (R_0, R_1) \in \mathbb{R}^2 : \\ R_1 \geq I(X;U)_\sigma , \\ R_0 + R_1 \geq I(XB;U)_\sigma \end{array} \right\} . \tag{7}$$

The proof for Theorem 1 is given in Subsection IV. The following corollaries immediately follow.

Corollary 2 (Quantum Common Information [23]). The coordination capacity without CR is

$$\mathsf{R}^{(0)}_{\text{2-node}}(\omega) = \min_{\sigma_{XUB}\in\mathscr{S}_{\text{2-node}}(\omega)} I(XB;U)_\sigma . \tag{8}$$

Corollary 3. The CR-assisted coordination capacity, i.e., with unlimited common randomness, is given by

$$\mathsf{R}^{(\infty)}_{\text{2-node}}(\omega) \triangleq \min_{\sigma_{XUB}\in\mathscr{S}_{\text{2-node}}(\omega)} I(X;U)_\sigma \tag{9}$$

We note that in order to achieve the CR-assisted capacity, a CR rate of $R_0 = I(U;B|X)$ is sufficient. If $B \equiv Y$ is classical, then we may substitute $U = Y$, which yields the capacity $\mathsf{R}^{(\infty)}_{\text{2-node}}(\omega) = I(X;Y)$, and it can be achieved with CR at rate $R_0 = H(Y|X)$ [27].

B. No-Communication Network

Consider a given quantum state ω_{ABC} that we wish to simulate. We now state our main result. Define the following set of c-q states. Let $\mathscr{S}_{NC}(\omega)$ be the set of all c-q states

$$\sigma_{UABC} = \sum_{u\in\mathcal{U}} p_U(u)\ |u\rangle\langle u|_U \otimes \theta^u_A \otimes \theta^u_B \otimes \theta^u_C \tag{10a}$$

such that

$$\sigma_{ABC} = \omega_{ABC} \tag{10b}$$

Given $U = u$, there is no correlation between A, B and C.

Theorem 4. The coordination capacity for the no-communication network described in Figure 2 is

$$\mathsf{C}_{\text{NC}}(\omega) = \inf_{\sigma_{UABC}\in\mathscr{S}_{NC}(\omega)} I(U;ABC)_\sigma \tag{11}$$

with the convention that an infimum over an empty set is $+\infty$.

Remark 1. Since the CR is classical, it cannot be used in order to create entanglement. Therefore, as Alice, Bob, and Charlie do not cooperate with one another, it is impossible to simulate entanglement. That is, we can only simulate separable states.

C. Broadcast Network

Consider a given c-q-q state $\omega_{XB_1B_2}$ that we wish to simulate. Define the following set of c-q-q states. Let $\mathscr{S}_{\text{2-BC}}(\omega)$ be the set of all c-q states

$$\sigma_{XUB_1B_2} = \sum_{\substack{(x,u)\in \\ \mathcal{X}\times\mathcal{U}}} p_{XU}(x,u)\, |x\rangle\langle x|_X \otimes |u\rangle\langle u|_U \otimes \theta^u_{B_1} \otimes \eta^u_{B_2}$$

such that

$$\sigma_{XB_1B_2} = \omega_{XB_1B_2} .$$

Note that given X, B_1, and B_2 are uncorrelated given $U = u$.

Theorem 5. The coordination capacity region of the broadcast channel in Figure 3 network is the set

$$\mathcal{R}_{\text{BC}}(\omega) = \bigcup_{\mathscr{S}_{BC}(\omega)} \left\{ \begin{array}{l} (R_0, R_1) \in \mathbb{R}^2 : \\ R_0 \geq I(X;U)_\sigma \\ R_0 + R_1 \geq I(XB_1B_2;U)_\sigma \end{array} \right\} . \tag{12}$$

Remark 2. Since Alice's encoding is classical, she cannot distribute entanglement. Therefore, as Bob 1 and Bob 2 do not cooperate with one another, it is impossible to simulate entanglement between Bob 1 and Bob 2. That is, we can only simulate states such that $\omega_{B_1B_2}$ is separable, as in the no-communication model (see Remark 1).

IV. Two Node Analysis

Consider the two node network in Figure 1. Our proof for Theorem 1 is based on quantum resolvability [28, 29].

Theorem 6 (see [28, 29]). Consider an ensemble, $\{p_X, \rho^x_A\}_{x\in\mathcal{X}}$, and a random codebook that consists of 2^{nR} independent sequence, $X^n(m)$, $m \in [2^{nR}]$, each is i.i.d.

$\sim p_X$. If $R > I(X;A)_\rho$, then for every $\delta > 0$ and sufficiently large n,

$$\mathbb{E}\left[\left\|\rho_A^{\otimes n} - \frac{1}{2^{nR}}\sum_{m=1}^{2^{nR}} \rho_{A^n}^{X^n(m)}\right\|_1\right] \leq \delta\,, \tag{13}$$

where $\rho_{A^n}^{x^n} \equiv \bigotimes_{k=1}^n \rho_A^{x_k}$, and the expectation is over all realizations of the random codebook.

A. Achievability proof

Assume (R_0, R_1) is in the interior of $\mathcal{R}_{\text{2-node}}(\omega)$. We need to construct a code that consists of an encoding channel $F(i|x^n, j)$ and a c-q decoding channel $\mathcal{D}_{IJ\to B^n}$, such that the error requirement in (4) holds.

By the definition of $\mathscr{S}_{\text{2-node}}(\omega)$, there exists a c-q state

$$\sigma_{UXB} = \sum_{u\in\mathcal{U}} p_U(u)\, |u\rangle\langle u|_U \otimes \sigma_{XB}^u \tag{14}$$

such that

$$\sigma_{XB}^u = \sum_{x\in\mathcal{X}} p_{X|U}(x|u)\, |x\rangle\langle x|_X \otimes \theta_B^u\,,\ u\in\mathcal{U} \tag{15}$$

$$\sigma_{XB} = \omega_{XB}\,, \tag{16}$$

$$R_1 \geq I(X;U)_\sigma\ ,\ R_0 + R_1 \geq I(XB;U)_\sigma\ . \tag{17}$$

Classical codebook generation: Select a random codebook $\mathscr{C} = \{u^n(i,j)\}$ by drawing $2^{n(R_0+R_1)}$ i.i.d. sequences according to the distribution $p_U^n(u^n) = \prod_{k=1}^n p_U(u_k)$. Reveal the codebook to Alice and Bob.

Let (i,j) be a pair of random indices, uniformly distributed over $[2^{nR_1}] \times [2^{nR_0}]$. Define the following PMF

$$\widetilde{P}_{X^nIJ}(x^n, i, j) \equiv \frac{1}{2^{n(R_0+R_1)}} p_{X|U}^n(x^n|u^n(i,j))\ . \tag{18}$$

Encoder: We define the encoding channel F as the conditional distribution above, i.e., $F = \widetilde{P}_{I|X^nJ}$.

Decoder: As Bob receives i from Alice, and the random element j, he prepares the output state $\mathcal{D}_{IJ\to B^n}(i,j) = \theta_{B^n}^{u^n(i,j)}$.

Error analysis: Let $\delta > 0$. The encoder sends $i \sim F(\cdot|x^n, j)$. Given $J = j$, by the classical resolvability theorem, Cuff [27] has shown that $R_1 \geq I(X;U)_\sigma$ guarantees

$$\mathbb{E}\left\|\widetilde{P}_{JX^n} - p_J \times p_X^n\right\|_1 \leq \delta \tag{19}$$

for sufficiently large n, where $\widetilde{P}_{JX^n}$ is as in (18). Recall that $\widetilde{P}_{JX^n}$ is random, since the codebook $\mathscr{C}$ is random. Hence, the expectation is over all realizations of $\mathscr{C}$. The resulting state is

$$\widehat{\rho}_{X^nB^n} = \frac{1}{2^{nR_0}} \sum_{j,x^n}\Big(p_X^n(x^n)\, |x^n\rangle\langle x^n|_{X^n} \otimes \sum_{i\in[2^{nR_1}]} \widetilde{P}_{I|X^nJ}(i|x^n,j)\theta_{B^n}^{u^n(i,j)}\Big) \tag{20}$$

According to (19), the probability distributions $\widetilde{P}_{J,X^n}$ and $p_J \times p_X^n$ are close on average. Then, let

$$\widehat{\tau}_{X^nB^n} \equiv \sum_{j,x^n} \widetilde{P}_{JX^n}(jx^n)\, |x^n\rangle\langle x^n|_{X^n} \otimes \sum_{i\in[2^{nR_1}]} \widetilde{P}_{I|X^nJ}(i|x^n,j)\theta_{B^n}^{u^n(i,j)}\,. \tag{21}$$

By (19), it follows that

$$\mathbb{E}\,\|\widehat{\tau}_{X^nB^n} - \widehat{\rho}_{X^nB^n}\|_1 \leq \delta\,. \tag{22}$$

Observe that

$$\begin{aligned}\widehat{\tau}_{X^nB^n} &= \sum_{i,j,x^n} \widetilde{P}_{IJX^n}(i,j,x^n)\, |x^n\rangle\langle x^n|_{X^n} \otimes \theta_{B^n}^{u^n(i,j)} \\ &= \frac{1}{2^{n(R_0+R_1)}} \sum_{i,j,x^n} p_{X|U}^n(x^n|u^n(i,j))\, |x^n\rangle\langle x^n|_{X^n} \otimes \theta_{B^n}^{u^n(i,j)} \\ &= \frac{1}{2^{n(R_0+R_1)}} \sum_{i,j} \sigma_{X^nB^n}^{u^n(i,j)}\end{aligned} \tag{23}$$

where the second equality is due to the definition of $\widetilde{P}$ in (18), and the last line follows from (15).

Thus, according to the quantum resolvability theorem, Theorem 6, when applied to the joint system XB, we have

$$\mathbb{E}\left\|\sigma_{XB}^{\otimes n} - \widehat{\tau}_{X^nB^n}\right\|_1 \leq \delta \tag{24}$$

for sufficiently large n. Therefore, by the triangle inequality,

$$\begin{aligned}&\mathbb{E}\left\|\omega_{XB}^{\otimes n} - \widehat{\rho}_{X^nB^n}\right\|_1 \\ &\leq \mathbb{E}\left\|\omega_{XB}^{\otimes n} - \widehat{\tau}_{X^nB^n}\right\|_1 + \mathbb{E}\|\widehat{\tau}_{X^nB^n} - \widehat{\rho}_{X^nB^n}\|_1 \leq 2\delta\end{aligned} \tag{25}$$

by (16), (22) and (24). ☐

B. Converse proof

Let (R_0, R_1) be an achievable rate pair. Then, there exists a sequence $\left(2^{nR_0}, 2^{nR_1}, n\right)$ coordination codes such that the joint quantum state $\widehat{\rho}_{X^nB^n}$ satisfies

$$\left\|\omega_{XB}^{\otimes n} - \widehat{\rho}_{X^nB^n}\right\|_1 \leq \varepsilon_n \tag{26}$$

where ε_n tends to zero as $n \to \infty$.

Fix an index $k \in \{1, \ldots, n\}$. By trace monotonicity [30], taking the partial trace over X_j, B_j, $j \neq k$, maintains the inequality. Thus,

$$\|\omega_{XB} - \widehat{\rho}_{X_kB_k}\|_1 \leq \varepsilon_n\,. \tag{27}$$

Then, by the AFW inequality [31],

$$\left|H(X^nB^n)_{\widehat{\rho}} - nH(XB)_\omega\right| \leq n\beta_n\,, \tag{28}$$

and

$$\left|H(X_kB_k)_{\widehat{\rho}} - H(XB)_\omega\right| \leq \beta_n\,, \tag{29}$$

for $k \in [n]$, where β_n tends to zero as $n \to \infty$. Therefore, by the triangle inequality,

$$\left|H(X^nB^n)_{\widehat{\rho}} - \sum_{k=1}^n H(X_kB_k)_{\widehat{\rho}}\right| \leq 2n\beta_n\,. \tag{30}$$

Now, we have

$$\begin{aligned} n(R_0+R_1) &\geq H(IJ) && (31)\\ &\geq I(X^nB^n; IJ)_{\widehat{\rho}} && (32)\end{aligned}$$

since the conditional entropy is nonnegative for classical and c-q states, and the CR element J is statistically independent of the source X^n. Furthermore, by entropy sub-additivity [30],

$$I(X^nB^n;IJ)_{\widehat{\rho}} \geq H(X^nB^n)_{\widehat{\rho}} - \sum_{k=1}^{n} H(X_kB_k|IJ)_{\widehat{\rho}}$$
$$\geq \sum_{k=1}^{n} I(X_kB_k;IJ)_{\widehat{\rho}} - 2n\beta_n \qquad (33)$$

where the last inequality follows from (30). Defining a time-sharing variable $K \sim \text{Unif}[n]$, this can be written as

$$R_0 + R_1 + 2\beta_n \geq I(X_KB_K;IJ|K)_{\widehat{\rho}} \qquad (34)$$

with respect to the extended state:

$$\widehat{\rho}_{KIJX_KB_K} = \frac{1}{n}\sum_{k=1}^{n} |k\rangle\langle k| \otimes \widehat{\rho}_{IJX_kB_k}\,. \qquad (35)$$

Observe that by (27) and the triangle inequality,

$$\|\omega_{XB} - \widehat{\rho}_{X_KB_K}\|_1 = \left\|\omega_{XB} - \frac{1}{n}\sum_{k=1}^{n}\widehat{\rho}_{X_kB_k}\right\|_1 \leq \varepsilon_n\,. \quad (36)$$

Thus, by the AFW inequality,

$$I(X_KB_K;K)_{\widehat{\rho}} = H(X_KB_K)_{\widehat{\rho}} - \frac{1}{n}\sum_{k=1}^{n} H(X_kB_k)_{\widehat{\rho}} \leq n\gamma_n\,,$$

where γ_n tends to zero. Together with (34), it follows that

$$R_0 + R_1 + 2\beta_n + \gamma_n \geq I(X_KB_K;IJK)_{\widehat{\rho}} \qquad (37)$$

By similar arguments,

$$R_1 + 2\beta_n + \gamma_n \geq I(X_K;IJ) \qquad (38)$$

To complete the converse proof, we identify U, X, and B with (I,J,K), X_K, and B_K, respectively. Observe that given (i,j,k), the joint state of X_K and B_K is $\left(\sum_{x_k\in\mathcal{X}} p_{X_k|IJ}(x_k|i,j)\,|x_k\rangle\langle x_k|_{X_K}\right)\otimes\rho_{B_k}^{(i,j)}$, where $p_{X^n|IJ}$ is the a posteriori probability distribution. Thus, there X and B are uncorrelated when conditioned on U, as required.

The bound on $|\mathcal{U}|$ follows by applying the Caratheodory theorem to the real-valued parameteric representation of density matrices, as in [32, App. B]. □

V. No-Communication Network Analysis

Consider the no-communication network in Figure 2, of a quantum state ω_{ABC}. To prove Theorem 4, we use similar tools. The achievability proof is straightforward, and it is thus omitted. Then, consider the converse part. Assume that R_0 is achievable. Therefore, there exists a sequence of $(2^{nR_0}, n)$ of coordination codes such that for sufficiently large values of n,

$$\left\|\widehat{\rho}_{A^nB^nC^n} - \omega_{ABC}^{\otimes n}\right\|_1 \leq \varepsilon_n\,, \qquad (39)$$

where $\varepsilon_n \to 0$ as $n \to \infty$.

Applying the chain rule,

$$nR_0 \geq H(J) \qquad (40)$$
$$\geq I(A^nB^nC^n;J)_{\widehat{\rho}} \qquad (41)$$
$$= \sum_{k=1}^{n} I(A_kB_kC_k;J|A^{k-1}B^{k-1}C^{k-1})_{\widehat{\rho}} \qquad (42)$$

According to similar considerations leading to (30), we have

$$I(A_kB_kC_k;A^{k-1}B^{k-1}C^{k-1})_{\widehat{\rho}} \leq \beta_n\,. \qquad (43)$$

Hence, by (42),

$$nR_0 \geq \sum_{k=1}^{n} I(A_kB_kC_k;JA^{k-1}B^{k-1}C^{k-1}) - n\beta_n$$
$$\geq \sum_{k=1}^{n} I(A_kB_kC_k;J) - n\beta_n$$
$$\geq n\left(\inf_{\sigma_{UABC}\in\mathscr{S}_{NC}(\omega)} I(U;ABC)_\sigma - 2\beta_n\right) \qquad (44)$$

taking $U \equiv J$, as the encoders are uncorrelated given J. □

VI. Summary and Discussion

We study coordination in three network models, two-node network simulating a c-q state, no-communication network simulating a separable state, and a broadcast network simulating c-q-q state. Our findings generalize classical results from [27] and [9], and also quantum results from [23]. The no-communication and broadcast networks can easily be extended to m encoders and decoders, respectively. The results are relevant for various applications, where the network nodes could represent classical-quantum sensors, computers performing a joint computation task, or players in a nonlocal game.

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References

[1] M. Mihaylov, Y.-A. Le Borgne, K. Tuyls, and A. Nowé, “Distributed cooperation in wireless sensor networks,” in *10th Int. Conf. Auton. Agents Multiagent Syst.-Volume 1*, 2011, pp. 249–256.

[2] C. Borcea, D. Iyer, P. Kang, A. Saxena, and L. Iftode, “Cooperative computing for distributed embedded systems,” in *Proc. 22nd Int. Conf. Distrib. Comput. Syst.*, 2002, pp. 227–236.

[3] C. Schieler and P. Cuff, “Rate-distortion theory for secrecy systemskaur2020nonlocality,” *IEEE Trans. Inf. Theory*, vol. 60, no. 12, pp. 7584–7605, 2014.

[4] T. A. Atif, M. Heidari, and S. S. Pradhan, “Faithful simulation of distributed quantum measurements with applications in distributed rate-distortion theory,” *IEEE Trans. Inf. Theory*, vol. 68, no. 2, pp. 1085–1118, 2022.

[5] H. Kaur and A. Kumar, “Nonlocality, entanglement, and randomness in different conflicting interest bayesian

games." *Quantum Inf. Comput.*, vol. 20, no. 11&12, pp. 901–934, 2020.
[6] G. P. Fettweis and H. Boche, "On 6G and trustworthiness," *Commun. ACM*, vol. 65, no. 4, pp. 48–49, 2022.
[7] R. Kuang and N. Bettenburg, "Shannon perfect secrecy in a discrete Hilbert space," in *IEEE Int. Conf. Quantum Comput. Eng. (QCE'2020)*, 2020, pp. 249–255.
[8] C. Wang and A. Rahman, "Quantum-enabled 6G wireless networks: Opportunities and challenges," *IEEE Wireless Commun.*, vol. 29, no. 1, pp. 58–69, 2022.
[9] P. W. Cuff, H. H. Permuter, and T. M. Cover, "Coordination capacity," *IEEE Trans. Inf. Theory*, vol. 56, no. 9, pp. 4181–4206, 2010.
[10] R. A. Chou, M. R. Bloch, and J. Kliewer, "Empirical and strong coordination via soft covering with polar codes," *IEEE Trans. Inf. Theory*, vol. 64, no. 7, pp. 5087–5100, 2018.
[11] E. Soljanin, "Compressing quantum mixed-state sources by sending classical information," *IEEE Trans. Inf. Theory*, vol. 48, no. 8, pp. 2263–2275, 2002.
[12] Z. Goldfeld, H. H. Permuter, and G. Kramer, "Duality of a source coding problem and the semi-deterministic broadcast channel with rate-limited cooperation," *IEEE Trans. Inf. Theory*, vol. 62, no. 5, pp. 2285–2307, 2016.
[13] M. A. Sohail, T. A. Atif, and S. S. Pradhan, "A new formulation of lossy quantum-classical and classical source coding based on a posterior channel," in *IEEE Int. Symp. Inf. Theory (ISIT'2023)*, 2023, pp. 743–748.
[14] H. M. Garmaroudi, S. S. Pradhan, and J. Chen, "Rate-limited quantum-to-classical optimal transport: A lossy source coding perspective," in *IEEE Int. Symp. Inf. Theory (ISIT'2023)*, 2023, pp. 1925–1930.
[15] H. H. Permuter, S. Shamai, and A. Somekh-Baruch, "Message and state cooperation in multiple access channels," *IEEE Trans. Inf. Theory*, vol. 57, no. 10, pp. 6379–6396, 2011.
[16] "Quantum channel simulation under purified distance is no more difficult than state splitting," `arXiv:2403.14416`, 2024.
[17] M. Berta, F. G. Brandão, M. Christandl, and S. Wehner, "Entanglement cost of quantum channels," *IEEE Trans. Inf. Theory*, vol. 59, no. 10, pp. 6779–6795, 2013.
[18] C. H. Bennett, I. Devetak, A. W. Harrow, P. W. Shor, and A. Winter, "The quantum reverse Shannon theorem and resource tradeoffs for simulating quantum channels," *IEEE Trans. Inf. Theory*, vol. 60, no. 5, pp. 2926–2959, 2014.
[19] S. Pirandola, S. L. Braunstein, R. Laurenza, C. Ottaviani, T. P. Cope, G. Spedalieri, and L. Banchi, "Theory of channel simulation and bounds for private communication," *Quantum Sci. Technol.*, vol. 3, no. 3, p. 035009, 2018.
[20] M. M. Wilde, "Entanglement cost and quantum channel simulation," *Phys. Rev. A*, vol. 98, no. 4, p. 042338, 2018.
[21] K. Li and Y. Yao, "Quantum broadcast channel simulation via multipartite convex splitting," *arXiv preprint arXiv:2112.04475*, 2021.
[22] L. Yu and V. Y. F. Tan, "Wyner's common information under Rényi divergence measures," *IEEE Trans. Inf. Theory*, vol. 64, no. 5, pp. 3616–3632, 2018.
[23] I. George, M.-H. Hsieh, and E. Chitambar, "One-shot bounds on state generation using correlated resources and local encoders," in *IEEE Int. Symp. Inf. Theory (ISIT'2023)*, 2023, pp. 96–101.
[24] A. Abeyesinghe, I. Devetak, P. Hayden, and A. Winter, "The mother of all protocols: Restructuring quantum information's family tree," *Proc. Roy. Soc. A: Math., Phys. Eng. Sci.*, vol. 465, no. 2108, pp. 2537–2563, 2009.
[25] K. Fang, X. Wang, M. Tomamichel, and M. Berta, "Quantum channel simulation and the channel's smooth max-information," *IEEE Trans. Inf. Theory*, vol. 66, no. 4, pp. 2129–2140, 2020.
[26] A. Wyner, "The common information of two dependent random variables," *IEEE Trans. Inf. Theory*, vol. 21, no. 2, pp. 163–179, 1975.
[27] P. Cuff, "Communication requirements for generating correlated random variables," in *2008 IEEE Int. Symp. Inf. Theory*, 2008, pp. 1393–1397.
[28] M. Hayashi, *Quantum Information Theory: Mathematical Foundation*. Springer, 2016.
[29] M. Tahmasbi and M. R. Bloch, "Framework for covert and secret key expansion over classical-quantum channels," *Phys. Rev. A*, vol. 99, p. 052329, May 2019.
[30] M. M. Wilde, *Quantum Information Theory*, 2nd ed. Cambridge University Press, 2017.
[31] A. Winter, "Tight uniform continuity bounds for quantum entropies: conditional entropy, relative entropy distance and energy constraints," *Commun. Math. Phys.*, vol. 347, pp. 291–313, 2016.
[32] U. Pereg, "Communication over quantum channels with parameter estimation," *IEEE Trans. Inf. Theory*, vol. 68, no. 1, pp. 359–383, 2022.

APPENDIX

Consider coordination in broadcast network, as in Figure 3 in the main text, of a classical-quantum-quantum state $\omega_{XB_1B_2}$. To prove the capacity theorem, Theorem 5, we use similar tools as in Section IV.

A. Achievability proof

Assume (R_0, R_1) is in the interior of $\mathcal{R}_{\text{BC}}(\omega)$. We need to construct a code that consists of an encoding channel $F(i|x^n, j)$ and a two c-q decoding channels $\mathcal{D}_{IJ\to B_1^n}$ and $\mathcal{D}_{IJ\to B_2^n}$,such that

$$\left\|\omega_{XB}^{\otimes n} - \frac{1}{2^{nR_0}}\sum_{j\in[2^{nR_0}]}\sum_{x^n\in\mathcal{X}^n} p_X^n(x^n)\,|x^n\rangle\langle x^n|_{X^n}\otimes \sum_{i\in[2^{nR_1}]} F(i|x^n,j)\mathcal{D}_{IJ\to B_1^n}(i,j)\otimes\mathcal{D}_{IJ\to B_2^n}(i,j)\right\|_1 \leq \varepsilon\,. \tag{45}$$

According to the definition of $\mathscr{S}_{\text{BC}}(\omega)$ (see Subsection III-C), there exists a c-q state $\sigma_{XUB_1B_2}$ that can be written as

$$\sigma_{XUB_1B_2} = \sum_{(x,u)\in\mathcal{X}\times\mathcal{U}} p_{X,U}(x,u)\,|x\rangle\langle x|_X \otimes |u\rangle\langle u|_U \otimes \theta_{B_1}^u\otimes\eta_{B_2}^u \tag{46}$$

and satisfy

$$\sigma_{XB_1B_2} = \omega_{XB_1B_2} \tag{47}$$

We will also consider conditioning on $U = u$, and denote

$$\sigma_{XB_1B_2}^u = \sum_{x\in\mathcal{X}} p_{X|U}(x|u)\,|x\rangle\langle x|_X\otimes\theta_{B_1}^u\otimes\eta_{B_2}^u\,. \tag{48}$$

Classical codebook generation: Select a random codebook $\mathscr{C}_{\text{BC}} = \{u^n(i,j)\}$ by drawing $2^{n(R_0+R_1)}$ i.i.d. sequences according to the distribution p_U^n. Reveal the codebook.

Encoder: Define the encoding channel as $F = \widetilde{P}_{I|X^nJ}$, where $\widetilde{P}_{X^nIJ}$ be a joint distribution as in (18).

Decoders: As Bob 1 and Bob 2 receive i from Alice, and the random element j, they prepare the following output states,

$$\mathcal{D}_{IJ\to B_1^n}^{(1)}(i,j) = \theta_{B_1}^{u^n(i,j)}\,, \tag{49}$$
$$\mathcal{D}_{IJ\to B_2^n}^{(2)}(i,j) = \eta_{B_2}^{u^n(i,j)}\,. \tag{50}$$

Error analysis: Let $\delta > 0$. The encoder sends $i \sim F(\cdot|x^n, j)$. As in Subsection IV-A, given j, if $R_1 \geq I(X;U)_\sigma$, then

$$\mathbb{E}\left\|\widetilde{P}_{JX^n} - p_J\times p_X^n\right\|_1 \leq \delta \tag{51}$$

for sufficiently large n. As $\widetilde{P}_{JX^n}$ depends on the random codebook $\mathscr{C}_{\text{BC}}$, the expectation is over all realizations of $\mathscr{C}_{\text{BC}}$. The resulting state is

$$\begin{aligned}\widehat{\rho}_{X^nB_1^nB_2^n} &= \frac{1}{2^{nR_0}}\sum_{j\in[2^{nR_0}]}\sum_{x^n\in\mathcal{X}^n}\Big(p_X^n(x^n)\,|x^n\rangle\langle x^n|_{X^n}\\ &\otimes\sum_{i\in[2^{nR_1}]}F(i|x^n,j)\mathcal{D}_{IJ\to B_1^n}(i,j)\otimes\mathcal{D}_{IJ\to B_2^n}(i,j)\Big)\\ &= \frac{1}{2^{nR_0}}\sum_{j,x^n}\Big(p_X^n(x^n)\,|x^n\rangle\langle x^n|_{X^n}\\ &\otimes\sum_{i\in[2^{nR_1}]}\widetilde{P}_{I|X^nJ}(i|x^n,j)\theta_{B_1}^{u^n(i,j)}\otimes\eta_{B_2}^{u^n(i,j)}\Big)\,.\end{aligned} \tag{52}$$

According to (51), the probability distributions $\widetilde{P}_{JX^n}$ and $p_J\times p_X^n$ are close on average. Then, let

$$\widehat{\tau}_{X^nB_1^nB_2^n} \equiv \sum_{j,x^n}\widetilde{P}_{JX^n}(j,x^n)\,|x^n\rangle\langle x^n|_{X^n}\otimes\sum_{i\in[2^{nR_1}]}\widetilde{P}_{I|X^nJ}(i|x^n,j)\theta_{B_1}^{u^n(i,j)}\otimes\eta_{B_2}^{u^n(i,j)}\,. \tag{53}$$

Then, it follows that

$$\mathbb{E}\left\|\widehat{\tau}_{X^nB_1^nB_2^n} - \widehat{\rho}_{X^nB_1^nB_2^n}\right\|_1 \leq \delta\,, \tag{54}$$

by (51). Observe that

$$\begin{aligned}&\widehat{\tau}_{X^nB_1^nB_2^n}\\ &= \sum_{i,j,x^n}\widetilde{P}_{IJX^n}(i,j,x^n)\,|x^n\rangle\langle x^n|_{X^n}\otimes\theta_{B_1}^{u^n(i,j)}\otimes\eta_{B_2}^{u^n(i,j)}\\ &= \frac{1}{2^{n(R_0+R_1)}}\sum_{i,j,x^n}p_{X|U}^n(x^n|u^n(i,j))\,|x^n\rangle\langle x^n|_{X^n}\\ &\otimes\theta_{B_1}^{u^n(i,j)}\otimes\eta_{B_2}^{u^n(i,j)}\\ &= \frac{1}{2^{n(R_0+R_1)}}\sum_{i,j}\sigma_{X^nB_1^nB_2^n}^{u^n(i,j)}\,,\end{aligned} \tag{55}$$

where the second equality is due to the definition of $\widetilde{P}$ in (18), and the last line follows from (48).

Thus, according to the quantum resolvability theorem 6, when applied to the joint system XB_1B_2, we have

$$\mathbb{E}\left\|\sigma_{XB_1^nB_2^n}^{\otimes n} - \widehat{\tau}_{X^nB_1^nB_2^n}\right\|_1 \leq \delta \tag{56}$$

for sufficiently large n. Therefore, by the triangle inequality,

$$\begin{aligned}&\mathbb{E}\left\|\omega_{XB_1B_2}^{\otimes n} - \widehat{\rho}_{X^nB_1^nB_2^n}\right\|_1\\ &\leq \mathbb{E}\left\|\omega_{XB_1^nB_2^n}^{\otimes n} - \widehat{\tau}_{X^nB_1^nB_2^n}\right\|_1 + \mathbb{E}\left\|\widehat{\tau}_{X^nB_1^nB_2^n} - \widehat{\rho}_{X^nB_1^nB_2^n}\right\|_1\\ &\leq 2\delta\end{aligned} \tag{57}$$

by (47), (54) and (56). □

B. Converse proof

Let (R_0, R_1) be an achievable coordination rate pair for the simulation of a c-q-q state $\omega_{XB_1B_2}$ in the broadcast setting. Then, there exists a sequence of $\left(2^{nR_0}, 2^{nR_1}, n\right)$ coordination codes such that the joint quantum state $\widehat{\rho}_{X^nB_1^nB_2^n}$ satisfies

$$\left\|\omega^{\otimes n}_{XB_1^nB_2^n} - \widehat{\rho}_{X^nB_1^nB_2^n}\right\|_1 \leq \varepsilon_n\,, \tag{58}$$

where ε_n tends to zero as $n \to \infty$. Fix an index $k \in \{1, \ldots, n\}$. By trace monotonicity [30], taking the partial trace over X_j, B_{1j}, B_{2j}, for $j \neq k$, maintains the inequality, thus

$$\|\omega_{XB_1B_2} - \widehat{\rho}_{X_kB_{1k}B_{2k}}\|_1 \leq \varepsilon_n \tag{59}$$

for $k \in [n]$. Hence,

$$\begin{aligned}&\left\|\widehat{\rho}_{XB_1^nB_2^n} - \bigotimes_{k=1}^{n} \widehat{\rho}_{X_kB_{1k}B_{2k}}\right\|_1 \\ &\leq \left\|\widehat{\rho}_{XB_1^nB_2^n} - \omega^{\otimes n}_{XB_1B_2}\right\|_1 + \prod_{k=1}^{n} \|\omega_{XB_1B_2} - \widehat{\rho}_{X_kB_{1k}B_{2k}}\|_1 \\ &\leq 2\varepsilon_n\,,\end{aligned} \tag{60}$$

where the second line follows from the triangle inequality and the last from (58)-(59). According to the AFW inequality [31] and the entropy chain rule, and since conditioning cannot increase entropy, we have

$$\left|H\left(X^nB_1^nB_2^n\right)_{\widehat{\rho}} - \sum_{k=1}^{n} H\left(X_kB_{1k}B_{2k}\right)_{\widehat{\rho}}\right| \leq n\beta_n \tag{61}$$

for $k \in [n]$, where β_n tends to zero as $n \to \infty$. Now, we also have

$$\begin{aligned}n(R_0 + R_1) &\geq H(IJ) \\ &\geq I(X^nB_1^nB_2^n; IJ)_{\widehat{\rho}}\,,\end{aligned} \tag{62}$$

since the conditional entropy is nonnegative for classical and c-q-q states, and the CR element J is statistically independent of the source X^n. Furthermore, by entropy sub-additivity [30],

$$\begin{aligned}&I(X^nB_1^nB_2^n; IJ)_{\widehat{\rho}} \\ &\geq H(X^nB_1^nB_2^n)_{\widehat{\rho}} - \sum_{k=1}^{n} H(X_kB_{1k}B_{2k}|IJ)_{\widehat{\rho}} \\ &\geq \sum_{k=1}^{n} I(X_kB_{1k}B_{2k}; IJ)_{\widehat{\rho}} - n\beta_n\end{aligned} \tag{63}$$

where the last inequality follows from (61).

Defining a time-sharing variable $K \sim \text{Unif}[n]$, this can be written as

$$R_0 + R_1 + \beta_n \geq I(X_KB_{1K}B_{2K}; IJ|K)_{\widehat{\rho}} \tag{64}$$

with respect to the extended state

$$\widehat{\rho}_{KIJX_kB_{1k}B_{2k}} = \frac{1}{n}\sum_{k=1}^{n} |k\rangle\langle k| \otimes \widehat{\rho}_{IJX_kB_{1k}B_{2k}}\,. \tag{65}$$

Observe that

$$\begin{aligned}&\|\omega_{XB_1B_2} - \widehat{\rho}_{X_KB_{1K}B_{2K}}\|_1 \\ &= \left\|\omega_{XB_1B_2} - \frac{1}{n}\sum_{k=1}^{n} \widehat{\rho}_{X_kB_{1k}B_{2k}}\right\|_1 \\ &\leq \varepsilon_n\end{aligned} \tag{66}$$

based on (59). Thus, by the AFW inequality,

$$\begin{aligned}&I(X_KB_{1K}B_{2K}; K)_{\widehat{\rho}} = \\ &\quad H(X_KB_{1K}B_{2K})_{\widehat{\rho}} - \frac{1}{n}\sum_{k=1}^{n} H(X_kB_{1k}B_{2k})_{\widehat{\rho}} \leq n\gamma_n\,,\end{aligned} \tag{67}$$

where γ_n tends to zero as $n \to \infty$. Together with (63), it implies

$$R_0 + R_1 + \beta_n + \gamma_n \geq I(X_KB_{1K}B_{2K}; IJK)_{\widehat{\rho}}\,. \tag{68}$$

By similar arguments,

$$R_1 + \beta_n + \gamma_n \geq I(X_K; IJ)\,. \tag{69}$$

To complete the converse proof, we identify U, X, and B_1B_2 with (I, J, K), X_K, and $B_{1K}B_{2K}$, respectively. Observe that given (i, j, k), the joint state of X_K and $B_{1K}B_{2K}$ is

$$\left(\sum_{x_k \in \mathcal{X}} p_{X_k|IJ}(x_k|i,j)\, |x_k\rangle\langle x_k|_{X_K}\right) \otimes \rho^{(i,j)}_{B_{1K}} \otimes \rho^{(i,j)}_{B_{2K}}\,, \tag{70}$$

where $p_{X^n|IJ}$ is the a posteriori probability distribution. Thus, there is no correlation between X, B_1, and B_2 when conditioned on U, as required. ☐