Communication with Unreliable Entanglement Assistance

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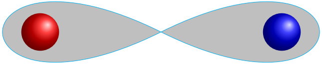
Joint Work with Christian Deppe and Holger Boche

Motivation: Entanglement

Entanglement resources are instrumental in a wide variety of quantum network frameworks:

- Physical-layer security (device-independent QKD, quantum repeaters) [Vazirani and Vidick 2014] [Yin et al. 2020][Pompili et al. 2021]



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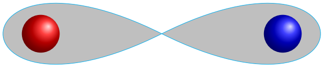

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- **Communication rate** [Bennett et al. 1999] [Hao et al. 2021]
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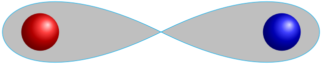

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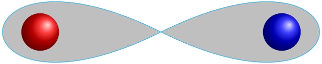
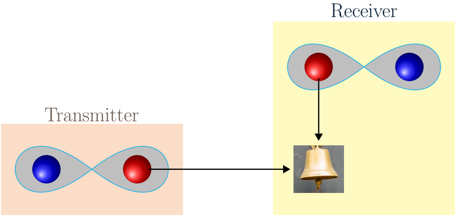
Unfortunately, entanglement is a fragile resource that is quickly degraded by decoherence effects.



Motivation: Entanglement (Cont.)

- In order to generate (heralded) entanglement in an optical communication system, the transmitter may prepare an entangled pair of photons locally, and then send one of them to the receiver.



Motivation: Entanglement (Cont.)

- In order to generate (heralded) entanglement in an optical communication system, the transmitter may prepare an entangled pair of photons locally, and then send one of them to the receiver.
- Such generation protocols are not always successful, as photons are easily absorbed before reaching the destination.

Motivation: Entanglement (Cont.)

- Therefore, practical systems require a back channel. In the case of failure, the protocol is to be repeated. The backward transmission may result in a delay, which in turn leads to a further degradation of the entanglement resources.

- We propose a new principle of operation: The communication system operates on a rate that is adapted to the status of entanglement assistance. Hence, feedback and repetition are not required.

Motivation: Entanglement (Cont.)

Alternative approaches for entanglement resources:

- Noisy entanglement [Zhuang, Zhu, and Shor 2017]
- Generate entanglement while the system is idle [Nötzel and DiAdamo, 2020]
- Entanglement distillation [Devetak and Winter, 2005]

Unreliable Resources in Classical Theory

Very partial list:

- Unreliable channel
  - outage capacity [Ozarow, Shamai, and Wyner 1994] [Ng et al. 2007]
  - automatic repeat request (ARQ) [Caire and Tuninetti 2001]
  - cognitive radio [Goldsmith et al. 2008]
  - connectivity [Simeone et al. 2012] [Tajer et al. 2021]

- Broadcast approach [Steiner and Shamai 2003] [Cohen, Médard, and Shamai 2022] (B.1)

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- Unreliable cooperation [Steinberg 2014]
  - cribbing encoders [Huleihel and Steinberg 2016]
  - conferencing decoders [Huleihel and Steinberg 2017] [Itzhak and Steinberg 2017] [P. and Steinberg 2020]

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The Fundamental Problem

Fundamental Problem: Noiseless Channel

Classical Bit-Pipe

The capacity of a classical noiseless bit channel is

$$1 \ \frac{\text{classical bit}}{\text{transmission}}$$

Holevo Bound

The classical capacity of a noiseless qubit channel is

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Fundamental Problem: Noiseless Channel + Assistance

Theorem

The classical *common-randomness* (CR) capacity of a noiseless bit-pipe is

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The classical *common-randomness* (CR) capacity of a noiseless bit-pipe is

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Theorem

The classical *entanglement-assisted* (EA) capacity of a noiseless qubit channel is

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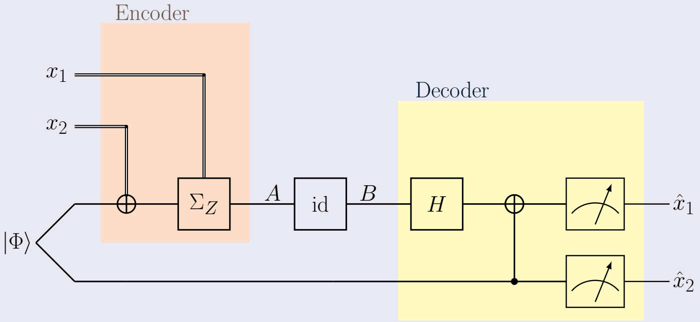
Fundamental Problem: Noiseless Channel + EA

Superdense Coding



Pauli

Fundamental Problem: Noiseless Channel + EA (Cont.)

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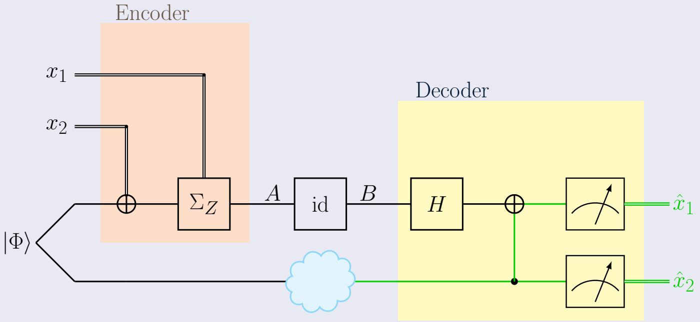

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Fundamental Problem: Noiseless Channel + EA (Cont.)

We consider transmission with unreliable EA:
The entangled resource may fail to reach Bob.

Extreme Strategies

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   - Excess rate: $R' = 0$

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2. Alice: Employ superdense encoder.

   Bob: If EA is **present**, employ superdense decoder.

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   If EA is absent, abort.

   - Guaranteed rate: $R = 0$
   - Excess rate: $R' = 2$

Fundamental Problem: Noiseless Channel + EA (Cont.)

Time Division

1st sub-block:

- Alice sends $(1-\lambda)n$ uncoded bits.
- Bob measures $(1-\lambda)n$ qubits without assistance.

2nd sub-block:

- Alice employs superdense encoding λn times.
- If EA is present, Bob decodes $2 \cdot \lambda n$ bits by superdense decoding.
- If EA is absent, Bob ignores λn qubits.

Fundamental Problem: Noiseless Channel + EA (Cont.)

Rates

- Guaranteed rate: $R = 1 - \lambda$
- Excess rate: $R' = 2\lambda$

⋆ Can we do better?

Main Contributions

- New principle of operation: communication over quantum channels with unreliable entanglement assistance.
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- New principle of operation: communication over quantum channels with unreliable entanglement assistance.
- Classical information:
  Alice sends classical messages to Bob
- Quantum information:
  Alice teleports a quantum state to Bob
- Time division, between entanglement-assisted and unassisted coding schemes, is optimal for a noiseless channel, but strictly sub-optimal for the depolarizing channel.

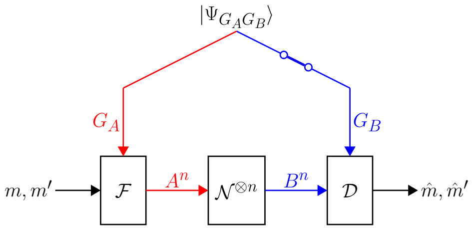
Outline

- Model
- Main Results

Classical Coding

Communication Scheme (1)

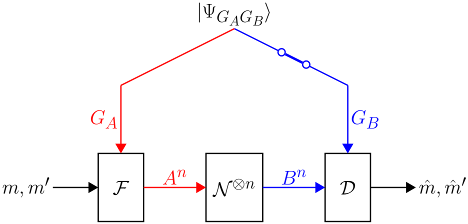
Alice chooses two messages, m and m'.



Classical Coding

Communication Scheme (2)

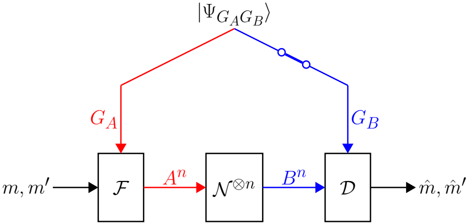
Input: Alice prepares $\rho_{A^n}^{m,m'} = \mathcal{F}^{m,m'}(\Psi_{G_A})$, and transmits A^n.
Output: Bob receives B^n.



Classical Coding

Decoding with Entanglement Assistance

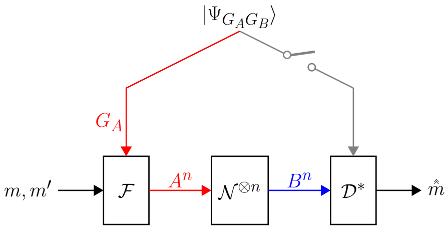
If EA is *present*, Bob performs a measurement $\mathcal{D}$ to estimate m, m'.



Classical Coding

Decoding without Assistance

If EA is absent, Bob performs a measurement $\mathcal{D}^*$ to estimate m alone.



Classical Coding (Cont.)

Error Probabilities

$$P^{(n)}_{e|m,m'} = 1 - \mathrm{Tr}\left[D_{m,m'}(\mathcal{N}^{\otimes n}_{A\to B} \otimes \mathrm{id})(\mathcal{F}^{m,m'} \otimes \mathrm{id})(\Psi_{G_A,G_B})\right]$$

$$P^{*(n)}_{e|m,m'} = 1 - \mathrm{Tr}\left[D^*_m \mathcal{N}^{\otimes n}_{A\to B} \mathcal{F}^{m,m'}(\Psi_{G_A})\right].$$

Classical Coding (Cont.)

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Capacity Region

- (R, R') is achievable with unreliable entanglement assistance if there exists a sequence of $(2^{nR}, 2^{nR'}, n)$ codes such that the error probabilities (with and without assistance) tend to zero as $n \to \infty$.

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Classical Coding (Cont.)

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- The classical capacity region $\mathcal{C}_{\mathrm{EA}*}(\mathcal{N})$ is the set of achievable rate pairs.

Quantum Coding

Quantum Coding

- Alice has a product state $\theta_M \otimes \xi_{\bar{M}}$ over Hilbert spaces of dimension $|\mathcal{H}_M| = 2^{nQ}$ and $|\mathcal{H}_{\bar{M}}| = 2^{n(Q+Q')}$
- She encodes by applying $\mathcal{F}_{G_A M \bar{M} \to A^n}$ to $\Psi_{G_A} \otimes \theta_M \otimes \xi_{\bar{M}}$, and transmits A^n.
- Bob receives ρ_{B^n}
- If EA is present, he applies $\mathcal{D}_{B^n G_B \to \tilde{M}}$.
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(Q, Q') is an achievable rate pair if there exists a sequence of $(2^{nQ}, 2^{nQ'}, n)$ codes such that

$$\left\|\xi_{\bar{M}} - \mathcal{D}(\rho_{B^n G_B})\right\|_1 \to 0 \quad \text{and} \quad \left\|\theta_M - \mathcal{D}^*(\rho_{B^n})\right\|_1 \to 0$$

as $n \to \infty$.

Related Work: Without Assistance

Let $\mathscr{N}_{A\to B}$ be quantum channel. Define the Holevo information

$$\chi(\mathscr{N}) = \max_{p_X(x), |\phi_A^x\rangle} I(X;B)_\rho$$

with $|\mathcal{X}| \leq |\mathcal{H}_A|^2$ and $\rho_{XB} \equiv \sum_{x\in\mathcal{X}} p_X(x) |x\rangle\langle x| \otimes \mathscr{N}_{A\to B}(\phi_A^x)$.

HSW Theorem
(Holevo 1998, Schumacher and Westmoreland 1997)

The classical capacity of a quantum channel $\mathscr{N}_{A\to B}$ without assistance satisfies

$$C_0(\mathscr{N}) = \lim_{k\to\infty} \frac{1}{k} \chi\left(\mathscr{N}^{\otimes k}\right)$$

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Related Work: Without Assistance (Cont.)

Let $\mathscr{N}_{A\to B}$ be quantum channel. Define

$$I_c(\mathscr{N}) = \max_{|\phi_{A_1 A}\rangle} I(A_1\rangle B)_\rho$$

with $\rho_{A_1 B} \equiv (\text{id} \otimes \mathscr{N}_{A\to B})(\phi_{A_1 A})$ and $|\mathcal{H}_{A_1}| = |\mathcal{H}_A|$.

Related Work: Without Assistance (Cont.)

Let $\mathscr{N}_{A\to B}$ be quantum channel. Define

$$I_c(\mathscr{N}) = \max_{|\phi_{A_1 A}\rangle} \left(-H(A_1|B)_\rho\right)$$

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LSD Theorem (Lloyd (1997), Shor (2002), and Devetak (2005))

The quantum capacity of a quantum channel $\mathscr{N}_{A\to B}$ is given by

$$Q_0(\mathscr{N}) = \lim_{k\to\infty} \frac{1}{k} I_c\left(\mathscr{N}^{\otimes k}\right)$$

Related Work: Entanglement Assistance

Theorem (Bennett, Shor, Smolin, and Thapliyal 1999)

The entanglement-assisted classical capacity of a quantum channel $\mathscr{N}_{A\to B}$ *is given by*

$$C_{EA}(\mathscr{N}) = \max_{|\phi_{A_1 A}\rangle} I(A_1;B)_\rho$$

with $\rho_{A_1 B} \equiv (id \otimes \mathscr{N})(\phi_{A_1 A})$.

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$$C_{EA}(\mathscr{N}) = \max_{|\phi_{A_1 A}\rangle} I(A_1; B)_\rho$$

and the entanglement-assisted quantum capacity is given by

$$Q_{EA}(\mathscr{N}) = \max_{|\phi_{A_1 A}\rangle} \frac{1}{2} I(A_1; B)_\rho$$

with $\rho_{A_1 B} \equiv (id \otimes \mathscr{N})(\phi_{A_1 A})$.

Outline

- Model
- Main Results

Main Results: Classical Capacity

Let $\mathscr{N}_{A\to B}$ be a quantum channel. Define

$$\mathcal{R}_{\mathsf{EA}*}(\mathscr{N}) = \bigcup_{p_X\,,\,|\phi_{A_0 A_1}\rangle\,,\,\mathcal{F}^{(x)}} \left\{ \begin{array}{rl} (R,R') \,:\, R \leq & I(X;B)_\rho \\ R' \leq & I(A_1;B|X)_\rho \end{array} \right\}$$

where the union is over the distributions p_X such that $|\mathcal{X}| \leq |\mathcal{H}_A|^2 + 1$, the pure states $|\phi_{A_0 A_1}\rangle$, and the quantum channels $\mathcal{F}^{(x)}_{A_0\to A}$, with

$$\rho_{XA_1A} = \sum_{x\in\mathcal{X}} p_X(x)|x\rangle\langle x| \otimes (\mathrm{id}\otimes\mathcal{F}^{(x)}_{A_0\to A})(|\phi_{A_1A_0}\rangle\langle\phi_{A_1A_0}|)\,,$$

$$\rho_{XA_1B} = (\mathrm{id}\otimes\mathscr{N}_{A\to B})(\rho_{XA_1A})\,.$$

Main Results: Classical Capacity (Cont.)

Theorem

The classical capacity region of a quantum channel $\mathscr{N}_{A\to B}$ with unreliable entanglement assistance satisfies

$$\mathcal{C}_{\mathsf{EA}*}(\mathscr{N}) = \bigcup_{k=1}^{\infty} \frac{1}{k} \mathcal{R}_{\mathsf{EA}*}(\mathscr{N}^{\otimes k}) \,.$$

Main Results: Classical Capacity (Cont.)

Classical "Superposition Coding"

- An auxiliary variable U is associated with the message m.

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- The entangled state $\phi_{A_0 A_1}$ is non-correlated with the messages, since the resources are pre-shared before communication takes place.
- Alice encodes the message m' using the encoding channel $\mathcal{F}^{(x)}_{A_0 \to A}$

Main Results: Classical Capacity (Cont.)

Corollary

For a noiseless qubit channel,

$$\mathcal{C}_{\mathsf{EA}*}(\mathscr{N}) = \bigcup_{0\leq\lambda\leq 1} \left\{ \begin{array}{lll} (R,R') : & R \leq & 1-\lambda \\ & R' \leq & 2\lambda \end{array} \right\}$$

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Proof: Achievability follows by time division. As for the converse part,

$$R \leq \frac{1}{n} I(X;B^n)_\omega \leq 1 - \frac{1}{n} H(B^n|X)_\omega$$

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Since $I(A;B)_\rho \leq 2H(B)_\rho$ in general, we have

$$R' \leq \frac{1}{n} I(A_1;B^n|X)_\omega \leq \frac{1}{n} \cdot 2H(B^n|X)_\omega$$

Set $\lambda \equiv \frac{1}{n} H(B^n|X)_\omega$. □

Main Results: Classical Capacity (Cont.)

Remark

The following tradeoff is observed:

- To maximize the unassisted rate, set an encoding channel $\mathcal{F}^{(x)}_{A_0 \to A}$ that outputs the pure state $|\psi^x_A\rangle$ that is optimal for the Holevo information, *i.e.*

$$\mathcal{F}^{(x)}(\varphi_{A_1 A_0}) = \varphi_{A_1} \otimes \psi^x_A$$
$$\Rightarrow (R, R') = (\chi(\mathscr{N}), 0)$$

▶ $\chi(\mathscr{N})$ **is achieved for an entanglement-breaking encoder**.

- For R' to achieve the entanglement-assisted capacity, set $\varphi_{A_0 A_1}$ as the entangled state that maximizes $I(A_1; B)_\rho$. Take $\mathcal{F}^{(x)} = \text{id}_{A_0 \to A}$. $\Rightarrow (R, R') = (0, C_{\text{EA}}(\mathscr{N}))$

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Example: Depolarizing Channel

Qubit depolarizing channel

$$\mathscr{N}(\rho) = (1-\varepsilon)\rho + \varepsilon\frac{\mathbb{1}}{2} \quad , \quad 0 \leq \varepsilon \leq 1$$

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$$\mathcal{N}(\rho) = (1-\varepsilon)\rho + \varepsilon\frac{\mathbb{1}}{2}$$
$$= \left(1 - \frac{3\varepsilon}{4}\right)\rho + \frac{\varepsilon}{4}\left(\Sigma_X\rho\Sigma_X + \Sigma_Y\rho\Sigma_Y + \Sigma_Z\rho\Sigma_Z\right)$$

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Example: Depolarizing Channel (Cont.)

Corner Points

- $\left[C(\mathscr{N}) = 1 - H_2\left(\frac{\varepsilon}{2}\right)\ ,\ 0\right]$ is achieved with $\left\{p_X = \left(\frac{1}{2}, \frac{1}{2}\right)\ ,\ \{|0\rangle, |1\rangle\}\right\}$
- $\left[0\ ,\ C_{\text{EA}}(\mathscr{N}) = 1 - H\left(1 - \frac{3\varepsilon}{4}, \frac{\varepsilon}{4}, \frac{\varepsilon}{4}, \frac{\varepsilon}{4}\right)\right]$
  is achieved with $|\Phi_{A_0 A_1}\rangle$ and $\mathcal{F}^{(x)} = \text{id}_{A_0 \to A}$.

Classical Mixture

Let $Z \sim \text{Bernoulli}(\lambda)$. Define $\mathcal{F}^{(x,z)}$ by $\mathcal{F}^{(x,0)}(\rho_A) = |x\rangle\langle x|$ and $\mathcal{F}^{(x,1)} = \text{id}$.
Plugging $\tilde{X} \equiv (X, Z)$, we obtain the time-division achievable region,

$$\mathcal{R}_{\text{EA*}}(\mathscr{N}) \supseteq \bigcup_{0 \le \lambda \le 1} \left\{ \begin{array}{rl} (R, R') : R \le & (1-\lambda)\, C(\mathscr{N}) \\ R' \le & \lambda C_{\text{EA}}(\mathscr{N}) \end{array} \right\}$$

Example: Depolarizing Channel (Cont.)

Quantum Superposition State

Define

$$|u_\beta\rangle \equiv \sqrt{1-\beta}\,|0\rangle \otimes |0\rangle + \sqrt{\beta}\,|\Phi\rangle \ .$$

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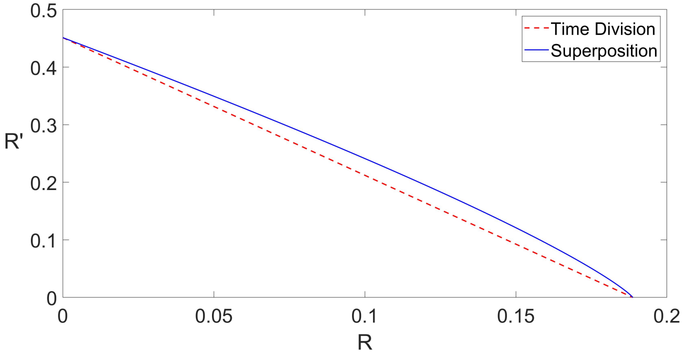
Set

$$|\phi_{A_0 A_1}\rangle \equiv \frac{1}{\| u_\beta \|}\,|u_\beta\rangle \quad , \quad p_X = \left(\frac{1}{2}, \frac{1}{2}\right) \quad , \quad \mathcal{F}^{(x)}(\rho) \equiv \Sigma_X^x \rho \Sigma_X^x$$

- For $\beta = 0$, the input state is $\mathcal{F}^{(x)}(|0\rangle\langle 0|) = |x\rangle\langle x|$, which achieves $C(\mathscr{N})$
- For $\beta = 1$, the parameter x chooses one of two bell states, achieving $C_{\text{EA}}(\mathscr{N})$

Example: Depolarizing Channel (Cont.)

Figure: Achievable rate regions for the depolarizing channel with $\varepsilon = \frac{1}{2}$.



Main Results: Quantum Capacity

Let $\mathscr{N}_{A\to B}$ be a quantum channel. Define

$$\mathscr{L}_{\text{EA}*}(\mathscr{N}) = \bigcup_{\varphi_{A_1 A_2 A}} \left\{ \begin{array}{l} (Q, Q') \ : \\ Q \leq \min\{I(A_1\rangle B)_\rho \, , \ H(A_1|A_2)_\rho\} \, , \\ Q + Q' \leq \frac{1}{2} I(A_2; B)_\rho \end{array} \right\}$$

where the union is over the states $\varphi_{AA_1A_2}$, with $\rho_{A_1A_2B} = (\text{id} \otimes \mathscr{N}_{A\to B})(\varphi_{A_1A_2A})$

Main Results: Quantum Capacity (Cont.)

Theorem

The quantum capacity region of a quantum channel $\mathscr{N}_{A\to B}$ with unreliable entanglement assistance satisfies

$$\mathcal{Q}_{\text{EA}*}(\mathscr{N}) = \bigcup_{k=1}^{\infty} \frac{1}{k} \mathscr{L}_{\text{EA}*}(\mathscr{N}^{\otimes k}) \,.$$

- The proof is based on the decoupling approach: By Uhlmann's theorem, if we can encode such that Alice and Bob's environments are in a product state, then there exists a decoding map such that $\mathcal{D} \circ \mathscr{N} \circ \mathcal{E} \approx \text{id}$.

Information-Theoretic Tools, Decoupling.

Summary and Concluding Remarks

- We considered communication over a quantum channel $\mathscr{N}_{A\to B}$, where Alice and Bob are provided with *unreliable* entanglement resources.

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- We considered communication over a quantum channel $\mathscr{N}_{A\to B}$, where Alice and Bob are provided with *unreliable* entanglement resources.
- Inspired by Steinberg's classical cooperation model, we developed a theory for **reliability by design** for entanglement-assisted point-to-point quantum communication systems.
- The quantum capacity formula has the following interpretation: Without assistance, A_2 behaves as a channel state system. The classical capacity formula resembles the superposition bound. A straightforward extension of our methods yields the capacity region of the broadcast channel with degraded message sets and one-sided entanglement assistance.

Summary and Concluding Remarks

- We considered communication over a quantum channel $\mathscr{N}_{A\to B}$, where Alice and Bob are provided with *unreliable* entanglement resources.
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- The quantum capacity formula has the following interpretation: Without assistance, A_2 behaves as a channel state system. The classical capacity formula resembles the superposition bound. A straightforward extension of our methods yields the capacity region of the broadcast channel with degraded message sets and one-sided entanglement assistance.
- In the future, it would be interesting to apply this methodology to other quantum information areas that rely on entanglement resources.

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