Communication over Quantum Channels with Parameter Estimation

Uzi Pereg

Technical University of Munich (TUM)

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Motivation

Quantum Communication and Information Theory

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- Progress in practice
    - Quantum key distribution for secure communication (307 km in optical fibers, 1200 km through space)
    - Computation power: Google's supremacy experiment

Motivation (Cont.)

Channels that depend on a random parameter

- Channel side information (CSI)
  - classical applications: cognitive radio in wireless systems, memory storage, digital watermarking, etc.

Motivation (Cont.)

Channels that depend on a random parameter

- Parameter estimation (RnS channel): the parameter sequence represents information that the decoder needs to reconstruct
  - recover digital watermarking + host data

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- Parameter estimation (RnS channel): the parameter sequence represents information that the decoder needs to reconstruct
  - recover digital watermarking + host data
  - multicast of control information on top of analog signal

Background: Random-Parameter Channel

Classical results with channel side information (CSI) at the encoder:

- Causal CSI [Shannon 1958]
- Strictly-causal CSI [Csiszár and Körner 1981]
- Non-causal CSI [Gel'fand and Pinsker 1980]

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Random-parameter classical-quantum channels

- Causal and Non-causal CSI [Boche, Cai, and Nötzel 2016]

Background: Random-Parameter Channels (Cont.)

Classical results with parameter estimation:

- Without CSI [Zhang, Vedantam, and Mitra 2011]
- Causal, strictly-causal CSI [Choudhuri, Kim, and Mitra 2013]
- Non-causal CSI [Sutivong, Chiang, Cover, and Kim 2005]
- CSI + feedback [Bross and Lapidoth 2018]

Main Contributions

We consider random-parameter quantum channels when the receiver reconstructs the parameter sequence with distortion

- Strictly-causal and causal CSI
  - Regularized formulas; single-letter for measurement channels
- Non-causal CSI
  - Regularized formula

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- Strictly-causal and causal CSI
  - Regularized formulas; single-letter for measurement channels
- Non-causal CSI
  - Regularized formula
- without CSI
  - Regularized formula; single-letter for entanglement-breaking channels
  - generalized Shor inequality

Outline

- Definitions
- Related Work
- Main Results

Quantum States

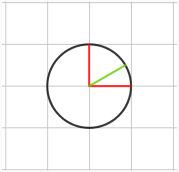
A pure quantum state $|\psi\rangle$ is a vector in the Hilbert space $\mathcal{H}_A$.

Qubit

For a qubit, $|\psi\rangle = |0\rangle$, $|1\rangle$, or

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \ , \ \text{with } |\alpha|^2 + |\beta^2| = 1$$

For $\alpha, \beta \in \mathbb{R}$:



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Entanglement

Systems A and B are entangled if $|\psi_{AB}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$

For example, $|\Phi_{AB}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$.

Quantum States (Cont.)

The state ρ_A of a quantum system A is an Hermitian, positive semidefinite, unit-trace density matrix over $\mathcal{H}_A$.

Measurement

A POVM (= positive-operator valued measure) is a set of positive semi-definite operators $\{\Lambda^j\}$ such that $\sum_j \Lambda^j = \mathbb{1}$. Born rule: the probability of the measurement outcome j is $p_A(j) = \mathrm{Tr}(\Lambda^j \rho_A)$.

Quantum Entropy and Mutual Information

Given ρ_{AB}, define

$$H(A)_\rho \equiv -\mathrm{Tr}(\rho_A \log \rho_A)$$

$$H(A|B)_\rho \equiv H(AB)_\rho - H(B)_\rho$$

$$I(A;B)_\rho \equiv H(A)_\rho + H(B)_\rho - H(AB)_\rho$$

Random Parameter Quantum Channel

A random-parameter quantum channel $\mathcal{N}_{SA\to B}$ is a linear, completely positive, trace preserving map corresponding to a quantum physical evolution:

$$\rho_A \xrightarrow{\mathcal{N}^{(s)}_{A\to B}} \rho_B$$

with

$$S \sim q(s)$$

Random Parameter Quantum Channel (Cont.)

Definition

A random-parameter quantum channel $\mathcal{N}_{SA\to B}$ is called **entanglement breaking** if for every state ρ_{AE} with arbitrary E, the output state $(\mathcal{N}^{(s)} \otimes \mathbb{1})(\rho_{AE})$ is separable, i.e.

$$(\mathcal{N}^{(s)} \otimes \mathbb{1})(\rho_{AE}) = \sum_y p_{Y|S}(y|s)\phi_B^{y,s} \otimes \phi_E^{y,s}$$

In particular, classical-quantum channels and quantum-classical channels are entanglement breaking.

Random Parameter Quantum Channel (Cont.)

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Definition

A quantum-classical channel is called a measurement channel.

Denote $\mathcal{M}_{SA\to Y}$.

Coding

Types of CSI

- None: Alice sends $\rho^m_{A^n}$

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- None: Alice sends $\rho^{m}_{A^n}$
- Strictly-causal: At time $i \in [1:n]$, Alice sends $\rho^{m,s_1,s_2,\ldots,s_{i-1}}_{A_i}$
- Causal: At time $i \in [1:n]$, Alice sends $\rho^{m,s_1,s_2,\ldots,s_i}_{A_i}$

Coding

Types of CSI

- None: Alice sends $\rho^{m}_{A^n}$
- Strictly-causal: At time $i \in [1:n]$, Alice sends $\rho^{m,s_1,s_2,\ldots,s_{i-1}}_{A_i}$
- Causal: At time $i \in [1:n]$, Alice sends $\rho^{m,s_1,s_2,\ldots,s_i}_{A_i}$
- Non-causal: Alice sends $\rho^{m,s^n}_{A^n}$

Coding with Strictly-causal CSI

Code

A $(2^{nR}, n)$ code with strictly-causal CSI at the encoder consists of a message set $[1 : 2^{nR}]$, a sequence of n consistent preparation maps $\mathcal{E}^{(i)}_{M,S^{i-1}\to A^i}$, for $i \in [1 : n]$, and a decoding POVM $\{\Lambda^{m,\hat{s}^n}_{B^n}\}_{m\in[1:2^{nR}],\hat{s}^n\in\widehat{\mathcal{S}}^n}$.

Denote the code by $(\mathcal{E}, \Lambda)$.

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Let $d : \mathcal{S} \times \widehat{\mathcal{S}} \to [0, d_{max}]$ be a given distortion measure, and

$$d^n(s^n, \hat{s}^n) \equiv \frac{1}{n}\sum_{i=1}^{n} d(s_i, \hat{s}_i)$$

Coding with Strictly-causal CSI (Cont.)

Probability of Error

$$P^{(n)}_{e|m} = \sum_{s^n \in \mathcal{S}^n} q^n(s^n) \mathrm{Tr}\left[(\mathbb{1} - \sum_{\hat{s}^n \in \widehat{S}^n} \Lambda^{m,\hat{s}^n}_{B^n}) \mathcal{N}^{(s^n)}_{A^n \to B^n} (\rho^{m,s^{n-1}}_{A^n}) \right]$$

with $\mathcal{N}^{(s^n)}_{A^n \to B^n} \equiv \bigotimes_{i=1}^{n} \mathcal{N}^{(s_i)}_{A \to B}$.

Distortion

$$\Delta^{(n)} = \mathbb{E} d^n(S^n, \hat{S}^n)$$

$$= \sum_{m,\hat{m},s^n,\hat{s}^n} d^n(s^n, \hat{s}^n) q^n(s^n) \cdot \frac{1}{2^{nR}} \mathrm{Tr}\left[\Lambda^{\hat{m},\hat{s}^n}_{B^n} \mathcal{N}^{(s^n)}_{A^n \to B^n} (\rho^{m,s^{n-1}}_{A^n}) \right]$$

Coding with Strictly-causal CSI (Cont.)

Capacity-Distortion Region

A $(2^{nR}, n, \varepsilon, D)$ code satisfies $P_{e|m}^{(n)} \leq \varepsilon \; \forall m$ and $\Delta^{(n)} \leq D$

A rate-distortion pair (R, D) is called achievable if $\forall \; \varepsilon, \delta > 0$ and $n \geq n_0$, there exists a $(2^{nR}, n, \varepsilon, D + \delta)$ code.

The capacity-distortion region $\mathcal{C}_{\text{s-c}}(\mathcal{N})$ is defined as the set of achievable pairs (R, D) with strictly-causal CSI.

Coding with Strictly-causal CSI (Cont.)

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$\Rightarrow$ The capacity-distortion function $C_{\text{s-c}}(\mathcal{N}, D)$

Coding with Strictly-causal CSI (Cont.)

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$\Rightarrow$ The capacity $C_{\text{s-c}}(\mathcal{N}, d_{max})$

Outline

- Definitions
- Related Work
- Main Results

Related Work: Without Parameters

Let $\mathcal{N}^0_{A\to B}$ be quantum channel without parameters. Define the Holevo information

$$\chi(\mathcal{N}^0) = \max_{p_X(x),|\phi^x_A\rangle} I(X;B)_\rho$$

with $|\mathcal{X}| \leq |\mathcal{H}_A|^2$ and $\rho_{XB} \equiv \sum_{x\in\mathcal{X}} p_X(x)|x\rangle\langle x| \otimes \mathcal{N}^0(\phi^x_A)$.

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HSW Theorem
(Holevo 1998, Schumacher and Westmoreland 1997)

The capacity of a quantum channel $\mathcal{N}^0_{A\to B}$ without parameters is given by

$$C(\mathcal{N}^0, d_{max}) = \lim_{n\to\infty} \frac{1}{n}\chi\left((\mathcal{N}^0)^{\otimes n}\right)$$

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If $\mathcal{N}^0_{A\to B}$ is entanglement-breaking, then $C(\mathcal{N}^0, d_{max}) = \chi(\mathcal{N}^0)$.

Related Work: Without Parameters (Cont.)

Additivity Conjecture (lasted until 2009)

For every pair of quantum channels $\mathcal{P}_{A_1 \to B_1}$ and $\Upsilon_{A_2 \to B_2}$,

$$\chi(\mathcal{P} \otimes \Upsilon) = \chi(\mathcal{P}) + \chi(\Upsilon)$$

and thus, the regularization in the HSW theorem can be removed.

Related Work: Without Parameters (Cont.)

Super-Additivity Property (Hastings 2009)

There exist quantum channels $\mathcal{P}_{A_1 \to B_1}$ and $\Upsilon_{A_2 \to B_2}$ such that

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Additivity for Entanglement-Breaking Channels (Shor 2002)

If $\mathcal{P}_{A_1\to B_1}$ is entanglement-breaking and $\Upsilon_{A_2\to B_2}$ is arbitrary, then

$$\chi(\mathcal{P}\otimes\Upsilon) = \chi(\mathcal{P}) + \chi(\Upsilon)$$

Back to our problem:

Random-Parameter Quantum Channels with Parameter Estimation

Outline

- Definitions
- Related Work
- Main Results

Main Results: No CSI

Alice has no knowledge on the parameters. Let

$$\mathcal{R}(\mathcal{N}) \triangleq \bigcup \left\{ \begin{array}{ll} (R,D)\ : & \\ \quad R \leq & I(X;B)_\rho \\ \quad D \geq & \sum_{s,\hat{s},x} q(s) p_X(x) \mathrm{Tr}(\Gamma^{\hat{s}}_{B|x} \mathcal{N}^{(s)}_{A\to B}(\phi^x_A)) d(s,\hat{s}) \end{array} \right\}$$

with $|\mathcal{X}| \leq |\mathcal{H}_A|^2 + 1$, where the union is over $p_X(x)$, state collection $\{|\phi^x_A\rangle\}$, and set of POVMs $\{\Gamma^{\hat{s}}_{B|x}\}$, with

$$\rho_{SXB} = \sum_{s,x} q(s) p_{X|S}(x|s) |s\rangle\langle s| \otimes |x\rangle\langle x| \otimes \mathcal{N}^{(s)}_{A\to B}(\phi^x_A)\,.$$

Main Results: No CSI (Cont.)

Theorem

The capacity-distortion region of a random-parameter quantum channel $\mathcal{N}_{SA\to B}$ without CSI is given by

$$\mathcal{C}(\mathcal{N}) = \bigcup_{n=1}^{\infty} \frac{1}{n}\mathcal{R}(\mathcal{N}^{\otimes n})$$

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Proof of 2nd part is not based on additivity $\quad \mathcal{R}(\mathcal{N}^{\otimes n}) \stackrel{?}{=} n\mathcal{R}(\mathcal{N})$

Analysis: No CSI, Entanglement-Breaking

Lemma (Shor Inequality)

Let $\mathcal{P}_{A_1\to B_1}$ and $\Upsilon_{A_2\to B_2}$ be quantum channels, where $\mathcal{P}_{A_1\to B_1}$ is entanglement breaking. Consider the classical-quantum states

$$\rho_{XA_1A_2} = \sum_{x\in\mathcal{X}} p_X(x)|x\rangle\langle x| \otimes \rho^x_{A_1A_2}$$

$$\rho_{XB_1B_2} \equiv (\mathbb{1} \otimes \mathcal{P} \otimes \Upsilon)(\rho_{XA_1A_2})$$

Then, there exists a classical-classical-quantum extension $\rho_{XYB_1B_2}$ such that

$$I(X;B_1,B_2)_\rho \leq I(X;B_1)_\rho + I(X,Y;B_2)_\rho \,.$$

Analysis: No CSI, Entanglement-Breaking

Lemma (Generalized Shor Inequality)

Let $\mathcal{N}_{SA\to B}$ be an entanglement-breaking random-parameter channel, and let $n \geq 2$. Consider the classical-quantum states

$$\rho_{XA^n} = \sum_{x\in\mathcal{X}} p_X(x)|x\rangle\langle x| \otimes \rho^x_{A_n}$$

$$\rho_{XB^n} \equiv \sum_{s^n\in\mathcal{S}^n} q^n(s^n)(\mathbb{1} \otimes \mathcal{N}^{(s^n)}_{A^n\to B^n})(\rho_{XA^n})$$

Then, there exists a classical-classical-quantum extension $\rho_{XY^{n-1}B^n}$ such that

$$I(X;B^n)_\rho \leq \sum_{i=1}^{n} I(X,Y^{i-1};B_i)_\rho \,.$$

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$$\rho_{XA^n} = \sum_{x\in\mathcal{X}} p_X(x)|x\rangle\langle x| \otimes \rho_{A_n}^x$$

$$\rho_{XB^n} \equiv \sum_{s^n\in\mathcal{S}^n} q^n(s^n)(\mathbb{1} \otimes \mathcal{N}_{A^n\to B^n}^{(s^n)})(\rho_{XA^n})$$

Then, there exists a classical-classical-quantum extension $\rho_{XY^{n-1}B^n}$ such that

$$I(X;B^n)_\rho \leq \sum_{i=1}^n I(X_i';B_i)_\rho \,.$$

Main Results: Strictly-Causal CSI

At time i, Alice knows $S_1, \ldots, S_{i-1}$. Let

$$\mathcal{R}_{\text{s-c}}(\mathcal{N}) \triangleq \bigcup \left\{ \begin{array}{rl} (R,D) \; : & \\ R \leq & I(Z,X;B)_\rho - I(Z;S|X) \\ D \geq & \sum\limits_{s,\hat{s},x,z} q(s) p_X(x) p_{Z|X,S}(z|x,s) \\ & \cdot \operatorname{Tr}(\Gamma^{\hat{s}}_{B|x,z} \mathcal{N}^{(s)}_{A\to B}(\phi^{z,x}_A)) d(s,\hat{s}) \end{array} \right\}$$

with $|\mathcal{X}| \leq |\mathcal{H}_A|^2 + 1$ and $|\mathcal{Z}| \leq |\mathcal{H}_A|^2 + |\mathcal{S}|$, where the union is over the set of all distributions $p_X(x) p_{Z|X,S}(z|x,s)$, state collection $\{|\phi^{z,x}_A\rangle\}$, and set of POVMs $\{\Gamma^{\hat{s}}_{B|x,z}\}$, with

$$\rho_{SZXB} = \sum_{s,z,x} q(s) p_X(x) p_{Z|X,S}(z|x,s) |s\rangle\langle s| \otimes |z\rangle\langle z| \otimes |x\rangle\langle x| \otimes \mathcal{N}^{(s)}_{A\to B}(\phi^{z,x}_A) .$$

Main Results: Strictly-Causal CSI (Cont.)

Theorem

The capacity-distortion region of a random-parameter quantum channel $\mathcal{N}_{SA\to B}$ with strictly-causal CSI at the encoder is given by

$$\mathcal{C}_{\text{s-c}}(\mathcal{N}) = \bigcup_{n=1}^{\infty} \frac{1}{n}\mathcal{R}_{\text{s-c}}(\mathcal{N}^{\otimes n})$$

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$$\mathcal{C}_{\text{s-c}}(\mathcal{N}) = \bigcup_{n=1}^{\infty} \frac{1}{n}\mathcal{R}_{\text{s-c}}(\mathcal{N}^{\otimes n})$$

For a random-parameter measurement channel, $\mathcal{C}_{\text{s-c}}(\mathcal{M}) = \mathcal{R}_{\text{s-c}}(\mathcal{M})$.

Main Results: Strictly-Causal CSI (Cont.)

Proof Key Ideas

- Extension of the methods of Choudhuri et al. (2013):
  a block Markov coding scheme where in each block we encode a fresh message and a compressed representation of the parameter sequence from the previous block.

Main Results: Strictly-Causal CSI (Cont.)

Proof Key Ideas

- Extension of the methods of Choudhuri et al. (2013):
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- Quantum packing lemma and classical covering lemma

Main Results: Strictly-Causal CSI (Cont.)

Proof Key Ideas

- Extension of the methods of Choudhuri et al. (2013):
  a block Markov coding scheme where in each block we encode a fresh message and a compressed representation of the parameter sequence from the previous block.
- Quantum packing lemma and classical covering lemma
- Gentle measurement lemma $\Rightarrow$ multiple measurements can be performed with negligible disturbance

Proof

Main Results: Causal CSI

At time i, Alice knows $S_1, \ldots, S_{i-1}, S_i$. Let

$$\mathcal{R}_{\text{caus}}(\mathcal{N}) \triangleq \bigcup \left\{ \begin{array}{rl} (R,D) : & \\ R \leq & I(Z,X;B)_\rho - I(Z;S|X) \\ D \geq & \sum_{s,\hat{s},x,z} q(s)p_X(x)p_{Z|X,S}(z|x,s) \\ & \cdot \operatorname{Tr}(\Gamma^{\hat{s}}_{B|x,z}\mathcal{N}^{(s)}_{A\to B}(\mathcal{F}^{(s)}_{K\to A}(\phi^{z,x}_K))d(s,\hat{s}) \end{array} \right\}$$

where the union is over $p_X(x)p_{Z|X,S}(z|x,s)$, state collection $\{|\phi^{z,x}_K\rangle\}$, quantum channels $\mathcal{F}^{(s)}_{K\to A}$, and set of POVMs $\{\Gamma^{\hat{s}}_{B|x,z}\}$, with

$$\rho_{SZXB} = \sum_{s,z,x} q(s)p_X(x)p_{Z|X,S}(z|x,s)|s\rangle\langle s| \otimes |z\rangle\langle z| \otimes |x\rangle\langle x| \otimes \mathcal{N}^{(s)}_{A\to B}(\mathcal{F}^{(s)}_{K\to A}(\phi^{z,x}_K)) .$$

Main Results: Causal CSI (Cont.)

Theorem

The capacity-distortion region of a random-parameter quantum channel $\mathcal{N}_{SA\to B}$ with causal CSI at the encoder is given by

$$\mathcal{C}_{\text{caus}}(\mathcal{N}) = \bigcup_{n=1}^{\infty} \frac{1}{n}\mathcal{R}_{\text{caus}}(\mathcal{N}^{\otimes n})$$

For a random-parameter measurement channel, $\mathcal{C}_{\text{caus}}(\mathcal{M}) = \mathcal{R}_{\text{caus}}(\mathcal{M})$.

Main Results: Causal CSI (Cont.)

Theorem

The capacity-distortion region of a random-parameter quantum channel $\mathcal{N}_{SA\to B}$ with causal CSI at the encoder is given by

$$\mathcal{C}_{\text{caus}}(\mathcal{N}) = \bigcup_{n=1}^{\infty} \frac{1}{n}\mathcal{R}_{\text{caus}}(\mathcal{N}^{\otimes n})$$

For a random-parameter measurement channel, $\mathcal{C}_{\text{caus}}(\mathcal{M}) = \mathcal{R}_{\text{caus}}(\mathcal{M})$.

In the proof, similar methods are applied to the virtual channel $\mathcal{L}_{SK\to B}$, where

$$\mathcal{L}^{(s)}_{K\to B}(\rho_K) \equiv \mathcal{N}^{(s)}_{A\to B}(\mathcal{F}^{(s)}_{K\to A}(\rho_K))$$

Main Results: Non-Causal CSI

Alice knows S^n a priori. Let

$$\mathcal{R}_{\text{n-c}}(\mathcal{N}) \triangleq \bigcup \left\{ \begin{array}{ll} (R,D) \; : & \\ \quad R \leq & I(X;B)_\rho - I(X;S) \\ \quad D \geq & \sum_{s,\hat{s},x} q(s) p_{X|S}(x|s) \\ & \cdot \mathrm{Tr}(\Gamma_{B|x}^{\hat{s}} \mathcal{N}_{A\to B}^{(s)}(\theta_A^{x,s})) d(s,\hat{s}) \end{array} \right\}$$

where the union is over $p_{X|S}(x|s)$, state collection $\{\theta_A^x\}$, and set of POVMs $\{\Gamma_{B|x}^{\hat{s}}\}$, with

$$\rho_{SXB} = \sum_{s,x} q(s) p_{X|S}(x|s) |s\rangle\langle s| \otimes |x\rangle\langle x| \otimes \mathcal{N}_{A\to B}^{(s)}(\theta_A^{x,s}) \,.$$

Main Results: Non-Causal CSI (Cont.)

Theorem

The capacity-distortion region of a random-parameter quantum channel $\mathcal{N}_{SA\to B}$ with non-causal CSI at the encoder is given by

$$\mathcal{C}_{\text{n-c}}(\mathcal{N}) = \bigcup_{n=1}^{\infty} \frac{1}{n} \mathcal{R}_{\text{n-c}}(\mathcal{N}^{\otimes n})$$

Even in the classical case, a single-letter characterization is an open problem.

Main Results: Without Estimation

Direct consequences that extend the results of Boche, Cai, and Nötzel (2016):

Corollary

The capacity of a random-parameter quantum channel $\mathcal{N}_{SA\to B}$ with strictly-causal CSI at the encoder is the same as without CSI.

* similar result for quantum feedback (clean channel from Bob to Alice) [Bowen 2004] and classical feedback for entanglement-breaking channels [Bowen and Nagarajan 2005]

Main Results: Without Estimation

Direct consequences that extend the results of Boche, Cai, and Nötzel (2016):

Corollary

The capacity of a random-parameter quantum channel $\mathcal{N}_{SA\to B}$ with causal CSI at the encoder is given by

$$C_{\text{caus}}(\mathcal{N}, d_{max}) = \lim_{k\to\infty} \frac{1}{k} \sup_{p_{X^k}(x^k),\, \mathcal{F}^{(s^k)}_{G^k\to A^k},\, |\phi^{x^k}_{G^k}\rangle} I(X^k; B^k)_\rho$$

with $\rho_{S^k X^k B^k} = \sum_{s^k,x^k} q^k(s^k) p_{X^k}(x^k) |s^k\rangle\langle s^k| \otimes |x^k\rangle\langle x^k| \otimes \mathcal{N}^{(s^k)}_{A^k\to B^k}\left(\mathcal{F}^{(s^k)}_{G^k\to A^k}(\phi^{x^k}_{G^k})\right)$

Main Results: Without Estimation

Direct consequences that extend the results of Boche, Cai, and Nötzel (2016):

Corollary

The capacity of a random-parameter quantum channel $\mathcal{N}_{SA\to B}$ with non-causal CSI at the encoder is given by

$$C_{\text{n-c}}(\mathcal{N}, d_{max}) = \lim_{k\to\infty} \frac{1}{k} \sup_{p_{X^k|S^k}(x^k|s^k)\,,\, \theta_{A^k}^{x^k,s^k}} [I(X^k;B^k)_\rho - I(X^k;S^k)]$$

with

$$\rho_{S^kX^kB^k} = \sum_{s^k,x^k} q^k(s^k)p_{X^k|S^k}(x^k|s^k)|s^k\rangle\langle s^k| \otimes |x^k\rangle\langle x^k| \otimes \mathcal{N}_{A^k\to B^k}^{(s^k)}(\theta_{A^k}^{x^k,s^k})$$

Summary

We derived regularized capacity-distortion formulas for quantum channels with parameter estimation in four scenarios.

- without CSI
  - Single-letter formula for entanglement-breaking channels
  - Alternative approach: generalized Shor inequality (instead of additivity)
- Strictly-causal and causal CSI
  - single-letter formula for measurement channels
  - Analysis: block Markov coding with binning + quantum packing lemma + gentle measurement
- Non-causal CSI
- Regularized capacity formulas without estimation

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Thank you

Email: uzi.pereg@tum.de