The Multiple-Access Channel With Entangled Transmitters

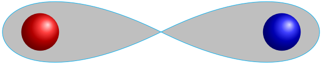
Uzi Pereg

ECE, Technion

Joint Work with Christian Deppe and Holger Boche

Background

Recently, there is growing interest in how quantum entanglement can assist classical networks.



Background

Recently, there is growing interest in how quantum entanglement can assist classical networks.

Single user:

- Entanglement resources do not increase the capacity of a classical channel [Bennett et al. 2002]

Background

Recently, there is growing interest in how quantum entanglement can assist classical networks.

Single user:

- Entanglement resources do not increase the capacity of a classical channel [Bennett et al. 2002]*

* ... but can increase the zero-error capacity [Leung et al., 2012]

Multi-user:

- multiple-access channel (MAC):
  entanglement resources between two transmitters can increase achievable rates!
  - pseudo-telepathy examples [Leditzky et al. 2020] [Seshadri et al. 2022] [Doolittle et al. 2022]
  - AVC Bell-violation example (with an adversary) [Nötzel 2020]

Multi-user:

- multiple-access channel (MAC):
  entanglement resources between two transmitters can increase achievable rates!
  - pseudo-telepathy examples [Leditzky et al. 2020] [Seshadri et al. 2022] [Doolittle et al. 2022]
  - AVC Bell-violation example (with an adversary) [Nötzel 2020]

Background (Cont.)

- non-signaling correlation can increase achievable rates
  - interference channels [Quek and Shor, 2017]
  - binary adder channel [Fawzi and Fermé, 2022]
- broadcast: entanglement resources between two receivers cannot increase achievable rates [Pereg et al. 2021]

** ... but can improve sensitivity in sensing [Zhang and Zhuang, 2021]

Background (Cont.)

- non-signaling correlation can increase achievable rates
  - interference channels [Quek and Shor, 2017]
  - binary adder channel [Fawzi and Fermé, 2022]

- broadcast: entanglement resources between two receivers cannot increase achievable rates [Pereg et al. 2021]

** ... but can improve sensitivity in sensing [Zhang and Zhuang, 2021]

Background (Cont.)

- non-signaling correlation can increase achievable rates
  - interference channels [Quek and Shor, 2017]
  - binary adder channel [Fawzi and Fermé, 2022]

- broadcast: entanglement resources between two receivers cannot increase achievable rates [Pereg et al. 2021]**

** ... but can improve sensitivity in sensing [Zhang and Zhuang, 2021]

Conferencing transmitters (very partial list)

- classical channels [Willems, 1983]
  - uncertainty [Maric et al., 2005]
  - AWGN [Wigger, 2008] [Bross et al., 2012]
  - jamming and secrecy [Wiese and Boche, 2014]
  - reliability [Steinberg, 2014] [Huleihel and Steinberg, 2017]
  - cloud radio-access network [Dikshtein et al. 2022]
  - ...
- c-q channels [Boche and Nötzel, 2014]

Main Contributions

We consider communication over a two-user classical MAC with entanglement resources shared between the transmitters, a priori before communication begins.

- the capacity region of the *general* MAC
- show that previous results can be obtained as a special case
- As opposed to the classical setting [Dueck 1978], the capacity region is remains the same, whether we consider a message-average or a maximal error criterion

Main Contributions

We consider communication over a two-user classical MAC with entanglement resources shared between the transmitters, a priori before communication begins.

- the capacity region of the *general* MAC
- show that previous results can be obtained as a special case
- As opposed to the classical setting [Dueck 1978], the capacity region is remains the same, whether we consider a message-average or a maximal error criterion
- Conferencing transmitters

Channel Model

Coding with Entanglement Resources

We consider a classical multiple-access channel, $P_{Y|X_1X_2}$.

Usually, in the classical model,

- Encoder 1 maps the message m_1 to a codeword x_1^n

  Encoder 2 maps the message m_2 to a codeword x_2^n

$$f_1 : \{1, \ldots, M_1\} \to \mathcal{X}_1^n$$
$$f_2 : \{1, \ldots, M_2\} \to \mathcal{X}_2^n$$

Coding with Entanglement Resources

We consider a classical multiple-access channel, $P_{Y|X_1X_2}$.

Usually, in the classical model,

- Encoder 1 maps the message m_1 to a codeword x_1^n

  Encoder 2 maps the message m_2 to a codeword x_2^n

$$f_1 : \{1, \dots, M_1\} \to \mathcal{X}_1^n$$
$$f_2 : \{1, \dots, M_2\} \to \mathcal{X}_2^n$$

- The codewords x_1^n, x_2^n are sent through n channel uses of $P_{Y|X_1,X_2}$

Coding with Entanglement Resources

We consider a classical multiple-access channel, $P_{Y|X_1X_2}$.

Usually, in the classical model,

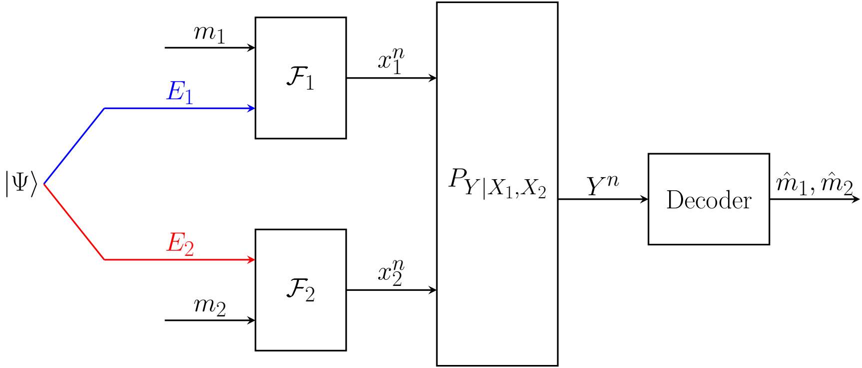
- Encoder 1 maps the message m_1 to a codeword x_1^n

  Encoder 2 maps the message m_2 to a codeword x_2^n

$$f_1 : \{1, \dots, M_1\} \to \mathcal{X}_1^n$$
$$f_2 : \{1, \dots, M_2\} \to \mathcal{X}_2^n$$

- The codewords x_1^n, x_2^n are sent through n channel uses of $P_{Y|X_1,X_2}$
- The decoder maps the channel output y^n to an estimation $(\widehat{m}_1, \widehat{m}_2)$

Here, the senders share an entangled state $\Psi_{E_1 E_2}$ a priori.



Coding with Entanglement Resources (Cont.)

Hence, an (M_1, M_2, n) code for the classical MAC with entangled transmitters consists of

- an entangled state $\Psi_{E_1 E_2}$ that is shared between the transmitters.
- two message sets $[M_1]$ and $[M_2]$
- Encoder 1 performs a measurement on E_1.

  Encoder 2 performs a measurement on E_2.

  Each has a collection of POVMs,

  $$\left\{ F_{x_1^n}^{(m_1)}, \; x_1^n \in \mathcal{X}_1^n \right\} \text{ and } \left\{ F_{x_2^n}^{(m_2)}, \; x_2^n \in \mathcal{X}_2^n \right\}$$

  one for each message.
- a decoding function $g : \mathcal{Y}^n \to [M_1] \times [M_2]$.

Coding with Entanglement Resources (Cont.)

Hence, an (M_1, M_2, n) code for the classical MAC with entangled transmitters consists of

- an entangled state $\Psi_{E_1 E_2}$ that is shared between the transmitters.
- two message sets $[M_1]$ and $[M_2]$
- Encoder 1 performs a measurement on E_1.

  Encoder 2 performs a measurement on E_2.

  Each has a collection of POVMs,

  $$\left\{ F_{x_1^n}^{(m_1)} \, , \; x_1^n \in \mathcal{X}_1^n \right\} \text{ and } \quad \left\{ F_{x_2^n}^{(m_2)} \, , \; x_2^n \in \mathcal{X}_2^n \right\}$$

  one for each message.
- a decoding function $g : \mathcal{Y}^n \to [M_1] \times [M_2]$.

Coding with Entanglement Resources (Cont.)

Hence, an (M_1, M_2, n) code for the classical MAC with entangled transmitters consists of

- an entangled state $\Psi_{E_1 E_2}$ that is shared between the transmitters.
- two message sets $[M_1]$ and $[M_2]$
- Encoder 1 performs a measurement on E_1.

  Encoder 2 performs a measurement on E_2.

  Each has a collection of POVMs,

  $$\left\{ F_{x_1^n}^{(m_1)} ,\ x_1^n \in \mathcal{X}_1^n \right\} \text{ and } \left\{ F_{x_2^n}^{(m_2)} ,\ x_2^n \in \mathcal{X}_2^n \right\}$$

  one for each message.
- a decoding function $g : \mathcal{Y}^n \to [M_1] \times [M_2]$.

Hence, an (M_1, M_2, n) code for the classical MAC with entangled transmitters consists of

- an entangled state $\Psi_{E_1 E_2}$ that is shared between the transmitters.
- two message sets $[M_1]$ and $[M_2]$
- Encoder 1 performs a measurement on E_1.

  Encoder 2 performs a measurement on E_2.

  Each has a collection of POVMs,

  $$\left\{ F_{x_1^n}^{(m_1)} ,\ x_1^n \in \mathcal{X}_1^n \right\} \text{ and } \left\{ F_{x_2^n}^{(m_2)} ,\ x_2^n \in \mathcal{X}_2^n \right\}$$

  one for each message.
- a decoding function $g : \mathcal{Y}^n \to [M_1] \times [M_2]$.

Coding with Entanglement Resources (Cont.)

Thus, the joint input distribution is

$$p(x_1^n, x_2^n | m_1, m_2) = \mathrm{Tr}\left[\left(F_{x_1^n}^{(m_1)} \otimes F_{x_2^n}^{(m_2)}\right) \Psi_{E_1 E_2}\right] .$$

The conditional probability of error given (m_1, m_2),

$$\Pr(\text{error}|m_1, m_2) =$$

$$\sum_{y^n : g(y^n) \neq (m_1, m_2)} \left[\sum_{x_1^n, x_2^n} p(x_1^n, x_2^n | m_1, m_2) P_{Y|X_1, X_2}^n (y^n | x_1^n, x_2^n)\right]$$

Coding with Entanglement Resources (Cont.)

The maximal probability of error is thus

$$P_e^{(n)} = \max_{m_1,m_2} \Pr(\text{error}|m_1, m_2)$$

Def: A rate pair (R_1, R_2) is achievable if there exists a sequence of (M_1, M_2, n) codes such that $\frac{1}{n}\log(M_k) \geq R_k$ for $k \in \{1, 2\}$, and

$$\lim_{n\to\infty} P_e^{(n)} = 0$$

Def: The capacity region $\mathcal{C}_{\text{ET}}$ of the classical MAC with entangled transmitters is defined as the closure of the set of achievable pairs (R_1, R_2).

Coding with Entanglement Resources (Cont.)

Remarks

- In communication, we often think of entanglement as the quantum version of common randomness (sharing a random key).
- Entanglement can generate common randomness.
- The capacity region with common randomness is the same as without it. That is, sharing a random key does not increase (asymptotically optimal) achievable rates in this setting.
- Entanglement improves achievable rates.

Remarks

- In communication, we often think of entanglement as the quantum version of common randomness (sharing a random key).
- Entanglement can generate common randomness.
- The capacity region with common randomness is the same as without it. That is, sharing a random key does not increase (asymptotically optimal) achievable rates in this setting.
- Entanglement improves achievable rates.

Main Results

Main Result

Define the rate regions

$$\mathcal{R}_{\mathsf{ET}}(P_{Y|X_1X_2}) = \bigcup_{p_{V_0}p_{V_1|V_0}p_{V_2|V_0}\,,\,\varphi_{A_1A_2}\,,\,L_1\otimes L_2} \left\{ \begin{array}{rl} (R_1,R_2)\ :\ R_1 & \le I(V_1;Y|V_0V_2) \\ R_2 & \le I(V_2;Y|V_0V_1) \\ R_1+R_2 & \le I(V_1V_2;Y|V_0) \end{array} \right\}$$

Main Result

Define the rate regions

$$\mathcal{R}_{\mathsf{ET}}(P_{Y|X_1X_2}) = \bigcup_{p_{V_0}p_{V_1|V_0}p_{V_2|V_0}\,,\,\varphi_{A_1A_2}\,,\,L_1\otimes L_2} \left\{ \begin{array}{rrl} (R_1,R_2)\ : & R_1 & \leq I(V_1;Y|V_0V_2) \\ & R_2 & \leq I(V_2;Y|V_0V_1) \\ & R_1+R_2 & \leq I(V_1V_2;Y|V_0) \end{array} \right\}$$

and

$$\mathcal{O}_{\mathsf{ET}}(P_{Y|X_1X_2}) = \bigcup_{p_{V_0V_1V_2}\,,\,\varphi_{A_1A_2}\,,\,L_1\otimes L_2} \left\{ \begin{array}{rrl} (R_1,R_2)\ : & R_1 & \leq I(V_1;Y|V_0V_2) \\ & R_2 & \leq I(V_2;Y|V_0V_1) \\ & R_1+R_2 & \leq I(V_1V_2;Y|V_0) \end{array} \right\}$$

Main Result

Define the rate regions

$$\mathcal{R}_{\mathsf{ET}}(P_{Y|X_1X_2}) = \bigcup_{V_1 -\!\!\circ\!\!- V_0 -\!\!\circ\!\!- V_2\,,\ \varphi_{A_1A_2}\,,\ L_1\otimes L_2} \left\{ \begin{array}{rl} (R_1,R_2)\ :\ R_1 & \leq I(V_1;Y|V_0V_2) \\ R_2 & \leq I(V_2;Y|V_0V_1) \\ R_1+R_2 & \leq I(V_1V_2;Y|V_0) \end{array} \right\}$$

and

$$\mathcal{O}_{\mathsf{ET}}(P_{Y|X_1X_2}) = \bigcup_{p_{V_0V_1V_2}\,,\ \varphi_{A_1A_2}\,,\ L_1\otimes L_2} \left\{ \begin{array}{rl} (R_1,R_2)\ :\ R_1 & \leq I(V_1;Y|V_0V_2) \\ R_2 & \leq I(V_2;Y|V_0V_1) \\ R_1+R_2 & \leq I(V_1V_2;Y|V_0) \end{array} \right\}$$

Main Result (Cont.)

The union is over the set of all

- entangled states $\varphi_{A_1 A_2}$
- classical auxiliary variables $(V_0, V_1, V_2) \sim p_{V_0} p_{V_1|V_0} p_{V_2|V_0}$
- collection of POVMs $\{L_1(x_1|v_0, v_1) \otimes L_2(x_2|v_0, v_2)\}$, for $v_0 \in \mathcal{V}_0$, $v_k \in \mathcal{V}_k$, $k \in \{1, 2\}$.

Given such a state, variables, and POVMs, $(V_0, V_1, V_2, X_1, X_2, Y)$ are distributed as

$$
\begin{aligned}
& p_{V_0}(v_0) p_{V_1|V_0}(v_1|v_0) p_{V_2|V_0}(v_2|v_0) \\
& \cdot \mathrm{Tr}\left[(L_1(x_1|v_0, v_1) \otimes L_2(x_2|v_0, v_2))\, \varphi_{A_1 A_2}\right] \\
& \cdot P_{Y|X_1, X_2}(y|x_1, x_2)
\end{aligned}
$$

Lemma

The union above is exhausted by auxiliary variables V_0, V_1, V_2 with $|\mathcal{V}_0| \leq 3$, $|\mathcal{V}_k| \leq 3(|\mathcal{X}_1||\mathcal{X}_2| + 2)$, $k = 1, 2$, and pure states $\varphi_{A_1 A_2} \equiv |\phi_{A_1 A_2}\rangle\langle\phi_{A_1 A_2}|$.

The proof of the lemma is based on purification, perturbation, and the support lemma (Fenchel-Eggleston-Carathéodory):

Any point in the convex closure of a connected compact set within $\mathbb{R}^d$ belongs to the convex hull of d points in the set.

Lemma

The union above is exhausted by auxiliary variables V_0, V_1, V_2 with $|\mathcal{V}_0| \leq 3$, $|\mathcal{V}_k| \leq 3(|\mathcal{X}_1||\mathcal{X}_2| + 2)$, $k = 1, 2$, and pure states $\varphi_{A_1 A_2} \equiv |\phi_{A_1 A_2}\rangle\langle\phi_{A_1 A_2}|$.

The proof of the lemma is based on purification, perturbation, and the support lemma (Fenchel-Eggleston-Carathéodory):

Any point in the convex closure of a connected compact set within $\mathbb{R}^d$ belongs to the convex hull of d points in the set.

Main Result (Cont.)

Lemma

The union above is exhausted by auxiliary variables V_0, V_1, V_2 with $|\mathcal{V}_0| \leq 3$, $|\mathcal{V}_k| \leq 3(|\mathcal{X}_1||\mathcal{X}_2| + 2)$, $k = 1, 2$, and pure states $\varphi_{A_1A_2} \equiv |\phi_{A_1A_2}\rangle\langle\phi_{A_1A_2}|$.

The proof of the lemma is based on purification, perturbation, and the support lemma (Fenchel-Eggleston-Carathéodory):

Any point in the convex closure of a connected compact set within $\mathbb{R}^d$ belongs to the convex hull of d points in the set.

Main Result (Cont.)

Remark

- To compute the region, one also needs to bound the dimension of A_1 and A_2

Remark

- To compute the region, one also needs to bound the dimension of A_1 and A_2
- It is **impossible**.

Main Result (Cont.)

Tsirelson conjecture: the set of correlation families

$$\left\{p(x_1, x_2|v_1, v_2) = \mathrm{Tr}(L_1(x_1|v_1) \otimes L_2(x_2|v_2))\right\}$$

with growing dimensions $\dim(\mathcal{H}_{A_k})$ is a closed set

Main Result (Cont.)

Tsirelson conjecture: the set of correlation families

$$\left\{p(x_1, x_2|v_1, v_2) = \mathrm{Tr}(L_1(x_1|v_1) \otimes L_2(x_2|v_2))\right\}$$

with growing dimensions $\dim(\mathcal{H}_{A_k})$ is a closed set

Theorem [Slofstra 2019] [Ji et al. 2021]

The Tsirelson conjecture is false.

Tsirelson conjecture: the set of correlation families

$$\Big\{p(x_1, x_2|v_1, v_2) = \mathrm{Tr}(L_1(x_1|v_1) \otimes L_2(x_2|v_2))\Big\}$$

with growing dimensions $\dim(\mathcal{H}_{A_k})$ is a closed set

Theorem [Slofstra 2019] [Ji et al. 2021]

The Tsirelson conjecture is false.

- Proof shows that there exists a family of refereed games such that it is undecidable to determine if $\exists$ a perfect strategy for a game in the family.

Main Result (Cont.)

Theorem

The capacity region of the classical MAC $P_{Y|X_1,X_2}$ with entangled transmitters is bounded by $\mathcal{R}_{\mathsf{ET}}(P) \subseteq \mathcal{C}_{\mathsf{ET}}(P) \subseteq \mathcal{O}_{\mathsf{ET}}(P)$. Furthermore,

$$\mathcal{C}_{\mathsf{ET}}(P) = \bigcup_{n=1}^{\infty} \frac{1}{n} \mathcal{R}_{\mathsf{ET}}(P^{\otimes n})$$

Main Result (Cont.)

Theorem

The capacity region of the classical MAC $P_{Y|X_1,X_2}$ with entangled transmitters is bounded by $\mathcal{R}_{\mathsf{ET}}(P) \subseteq \mathcal{C}_{\mathsf{ET}}(P) \subseteq \mathcal{O}_{\mathsf{ET}}(P)$. Furthermore,

$$\mathcal{C}_{\mathsf{ET}}(P) = \bigcup_{n=1}^{\infty} \frac{1}{n} \mathcal{R}_{\mathsf{ET}}(P^{\otimes n})$$

In the achievability proof:

- Prepare $\varphi_{A_1 A_2}^{\otimes n}$
- Generate i.i.d. v_0^n, $v_1^n(m_1)$, $v_2(m_2)$
- Measure x_k^n using POVM $\bigotimes_{i=1}^{n} L_k(x_{k,i} \mid v_{0,i}, v_{k,i}(m_k))$, $k \in \{1, 2\}$

Our lemma and theorem imply the following.

Corollary (lower bound)

A rate pair (R_1, R_2) is achievable with entanglement at rate θ_E if

$$R_1 \leq I(V_1; Y|V_0V_2)\,,\ R_2 \leq I(V_2; Y|V_0V_1)\,,\ R_1 + R_2 \leq I(V_1V_2; Y|V_0)$$

for $|\phi_{A_1A_2}\rangle$ with $H(A_1)_\phi = H(A_2)_\phi \leq \theta_E$,

some distribution of (V_0, V_1, V_2) and measurements $L_1 \otimes L_2$.

Main Result (Cont.)

Our lemma and theorem imply the following.

Corollary (lower bound)

A rate pair (R_1, R_2) is achievable with entanglement at rate θ_E if

$$R_1 \leq I(V_1; Y|V_0V_2)\,,\ R_2 \leq I(V_2; Y|V_0V_1)\,,\ R_1 + R_2 \leq I(V_1V_2; Y|V_0)$$

for $|\phi_{A_1A_2}\rangle$ with $H(A_1)_\phi = H(A_2)_\phi \leq \theta_E$,
some distribution of (V_0, V_1, V_2) and measurements $L_1 \otimes L_2$.

Proof: Every pure state $|\phi_{AB}\rangle$ has a Schmidt decomposition,

$$|\phi_{AB}\rangle = \sum_{x\in\mathcal{X}} \sqrt{p_X(x)}\,|\psi'_x\rangle \otimes |\psi''_x\rangle$$

with $|\mathcal{X}| \leq \min\{\dim(\mathcal{H}_A)\,,\ \dim(\mathcal{H}_B)\}$

Example 1 [Leditzky et al. 2020]

Consider a refereed game:

A referee selects questions (v_1, v_2) (e.g., uniformly),

- Player 1 responds with w_1, and
- Player 2 responds with w_2

Example 1 [Leditzky et al. 2020]

Consider a refereed game:

A referee selects questions (v_1, v_2) (e.g., uniformly),

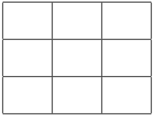
- Player 1 responds with w_1, and
- Player 2 responds with w_2

They win if $(v_1, w_1, v_2, w_2) \in \mathscr{G}$.

Peres and Mermin (1990) introduced the **magic square game**.

A referee selects (i, j) uniformly at random, and asks:

- Alice to fill ith row with even parity, and
- Bob to fill jth column with odd parity.



Example 1 (Cont.) [Leditzky et al. 2020]

Peres and Mermin (1990) introduced the magic square game.

A referee selects (i, j) uniformly at random, and asks:

- Alice to fill ith row with even parity, and
- Bob to fill jth column with odd parity.

For example, for $(i, j) = (2, 1)$:

0	1	1

TECHNION | Helen Diller Quantum Center

Example 1 (Cont.) [Leditzky et al. 2020]

Peres and Mermin (1990) introduced the magic square game.

A referee selects (i, j) uniformly at random, and asks:

- Alice to fill ith row with even parity, and
- Bob to fill jth column with odd parity.

For example, for $(i, j) = (2, 1)$:

0		
1	1	1
0		

Example 1 (Cont.) [Leditzky et al. 2020]

Peres and Mermin (1990) introduced the magic square game.

A referee selects (i, j) uniformly at random, and asks:

- Alice to fill ith row with even parity, and
- Bob to fill jth column with odd parity.

For example, for $(i, j) = (2, 1)$:

0		
1	1	1
0		

They win if they agree on the overlapping cell.

Example 1 (Cont.) [Leditzky et al. 2020]

Peres and Mermin (1990) introduced the magic square game.

A referee selects (i, j) uniformly at random, and asks:

- Alice to fill ith row with even parity, and
- Bob to fill jth column with odd parity.

For example, for $(i, j) = (2, 1)$:

0		
0	1	1
0		

They win if they agree on the overlapping cell.

Example 1 (Cont.) [Leditzky et al. 2020]

Peres and Mermin (1990) introduced the magic square game.

A referee selects (i, j) uniformly at random, and asks:

- Alice to fill ith row with even parity, and
- Bob to fill jth column with odd parity.

For example, for $(i, j) = (2, 1)$:

0		
1	1	1
0		

They win if they agree on the overlapping cell.

Example 1 (Cont.) [Leditzky et al. 2020]

Peres and Mermin (1990) introduced the magic square game.

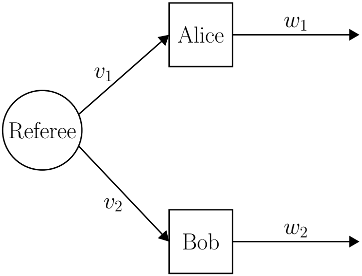
A referee selects (i, j) uniformly at random, and asks:

- Alice to fill ith row with even parity, and
- Bob to fill jth column with odd parity.

0	0	0
0	1	1
1	0	?

Example 1 (Cont.) [Leditzky et al. 2020]

This is an example of a refereed game.

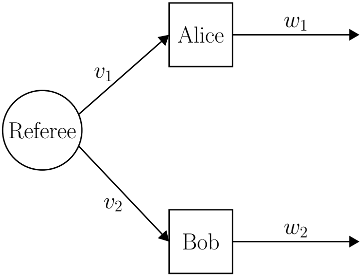


Using classical strategies:

- It is impossible to win with deterministic strategies.

Example 1 (Cont.) [Leditzky et al. 2020]

This is an example of a refereed game.

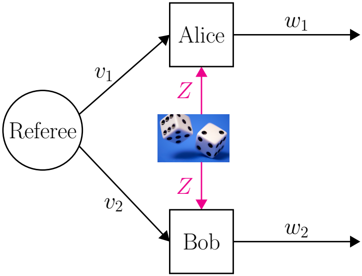


Using classical strategies:

- It is impossible to win with deterministic strategies.
- Using random strategies, $\Pr(\text{winning}) \leq \frac{8}{9}$

Example 1 (Cont.) [Leditzky et al. 2020]

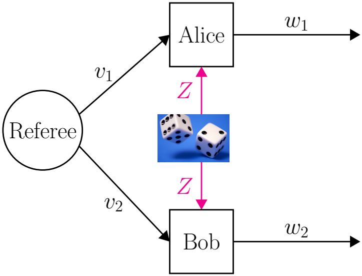
This is an example of a refereed game.



Using classical strategies:

Example 1 (Cont.) [Leditzky et al. 2020]

This is an example of a refereed game.

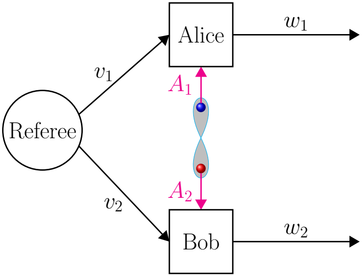


Using classical strategies:

- Given correlation (shared key), $\Pr(\text{winning}) \leq \frac{8}{9}$

Example 1 (Cont.) [Leditzky et al. 2020]

This is an example of a refereed game.



Using quantum strategies:

- Given entanglement correlation, $\Pr(\text{winning}) = 1$

Example 1 (Cont.) [Leditzky et al. 2020]

An equivalent formulation of the magic square game with ± 1:

$+1$	$+1$	$+1$
$+1$	-1	-1
-1	$+1$	?

such that the row and column products are $+1$ and -1, respectively.

Example 1 (Cont.) [Leditzky et al. 2020]

The following quantum strategy wins the game with probability 1:

- Prepare $|\psi_{A_1B_1A_2B_2}\rangle = |\Phi_{A_1B_1}\rangle \otimes |\Phi_{A_2B_2}\rangle$, a priori. Hence,

$$|\psi_{A_1A_2B_1B_2}\rangle = \frac{1}{2}(|00\rangle|00\rangle + |01\rangle|01\rangle + |10\rangle|10\rangle + |11\rangle|11\rangle)$$

- Measure in the bases as in the table below,

$X \otimes \mathbb{1}$	$\mathbb{1} \otimes X$	$X \otimes X$
$-X \otimes Z$	$-Z \otimes X$	$Y \otimes Y$
$\mathbb{1} \otimes Z$	$Z \otimes \mathbb{1}$	$Z \otimes Z$

simultaneously. This is possible because the operators in each row/column commute.

Example 1 (Cont.) [Leditzky et al. 2020]

Magic-Square Channel

Define $P_{Y|X_1,X_2}$ with

$$\mathcal{X}_1 = \mathcal{X}_2 = \{1,2,3\} \times \{0,1\}^3$$
$$\mathcal{Y} = \{1,2,3\}^2$$

such that given $X_1 = (V_1, W_1)$ and $X_2 = (V_2, W_2)$,

$$Y = (V_1, V_2) \quad \text{if } (X_1, X_2) = (V_1, W_1, V_2, W_2) \in \mathscr{G}_{MS}$$

$$Y \sim \mathrm{Uniform}(\{1,2,3\}^2) \quad \text{otherwise}$$

where $\mathscr{G}_{MS}$ is the winning set for the magic-square game.

Example 1 (Cont.) [Leditzky et al. 2020]

Without entanglement resources [Seshadri et al. 2022], the sum-rate is bounded by

$$R_1 + R_2 \leq 3.02$$

Example 1 (Cont.) [Leditzky et al. 2020]

Without entanglement resources [Seshadri et al. 2022], the sum-rate is bounded by

$$R_1 + R_2 \leq 3.02$$

Given entanglement between the transmitters, $R_1 + R_2 = 2\log(3) \approx 3.17$ is achievable [Leditzky et al. 2020]

Example 1 (Cont.) [Leditzky et al. 2020]

We can also obtain the capacity region from our theorem:

$$\mathcal{C}_{\mathsf{ET}} = \left\{ (R_1, R_2) : \begin{array}{c} R_1 \leq \log(3) \\ R_2 \leq \log(3) \end{array} \right\}$$

as we set

- entangled state: $\Phi_{A_1' A_2'} \otimes \Phi_{A_1'' A_2''}$
- auxiliary variables: $V_0 = \emptyset$, $(V_1, V_2) \sim \text{Uniform}(\{1,2,3\}^2)$
- measure (W_1, W_2) as in the magic-square game and transmit $X_k = (W_k, V_k)$.

Since (X_1, X_2) win the game, we have $Y = (V_1, V_2)$.

We can also obtain the capacity region from our theorem:

$$\mathcal{C}_{\mathsf{ET}} = \left\{ (R_1, R_2) : \quad \begin{array}{l} R_1 \le \log(3) \\ R_2 \le \log(3) \end{array} \right\}$$

as we set

- entangled state: $\Phi_{A_1' A_2'} \otimes \Phi_{A_1'' A_2''}$ (requires $\theta_E = 2$)
- auxiliary variables: $V_0 = \emptyset$, $(V_1, V_2) \sim \mathrm{Uniform}(\{1,2,3\}^2)$
- measure (W_1, W_2) as in the magic-square game and transmit $X_k = (W_k, V_k)$.

Since (X_1, X_2) win the game, we have $Y = (V_1, V_2)$.

Example 2 [Leditzky et al. 2020]

Slofstra and Vidick (2018) presented a linear equation game. Consider a $K \times N$ equation system, $\mathbf{H}\mathbf{u} = \mathbf{b}$, over $\mathrm{GF}(2)$.

A referee selects an equation $k \in \{1, \ldots, K\}$ and a variable $j \in \{1, \ldots, N\}$, uniformly at random, and sends to players.

- Player 1 gives $\mathbf{u}_1 \in \{0,1\}^N$ that satisfies Equation $\#k$
- Player 2 give $u_2[j] \in \{0,1\}$

They win if $\mathbf{H}[k,j] = 0$ or $u_1[j] = u_2[j]$.

The MAC $P_{Y|X_1,X_2}$ is defined in a similar manner.

Example 2 (Cont.) [Leditzky et al. 2020]

Theorem [Slofstra and Vidick 2018]

There exist linear equation systems such that

- quantum strategies can outperform classical strategies
- the minimal entanglement dimension to win w.p. $1 - e^{-T}$ satisfies

$$Ce^{T/6} \leq d_{E,\min} \leq C' e^{T/2}$$

for $T > 0$, where $C, C' > 0$ are constants.

It follows that the game can be won with certainty for $d_E \to \infty$, but not for $d_E < \infty$.

Achieving the capacity region

$$\mathcal{C}_{\text{ET}} = \left\{ (R_1, R_2) : \begin{array}{l} R_1 \leq \log(K) \\ R_2 \leq \log(N) \end{array} \right\}$$

requires infinite amount of entanglement [Leditzky et al. 2020]

Furthermore, we observe that the following region is achievable with entanglement rate $\theta_E = \frac{1}{2}T + \log(C')$,

$$R_1 \leq (1 - 4e^{-T}) \log(K) - 2(1 + e^{-T}) h_2 \left(\frac{e^{-T}}{1 + e^{-T}} \right),$$

$$R_2 \leq (1 - 4e^{-T}) \log(N) - 2(1 + e^{-T}) h_2 \left(\frac{e^{-T}}{1 + e^{-T}} \right)$$

for all $T > 0$, where $h_2(p)$ is the binary entropy function.

Example 2 (Cont.) [Leditzky et al. 2020]

Achieving the capacity region

$$\mathcal{C}_{\mathrm{ET}} = \left\{ (R_1, R_2) : \begin{array}{l} R_1 \leq \log(K) \\ R_2 \leq \log(N) \end{array} \right\}$$

requires infinite amount of entanglement [Leditzky et al. 2020]

Furthermore, we observe that the following region is achievable with entanglement rate $\theta_E = \frac{1}{2}T + \log(C')$,

$$R_1 \leq \left(1 - 4e^{-T}\right)\log(K) - 2(1 + e^{-T})h_2\left(\frac{e^{-T}}{1 + e^{-T}}\right),$$

$$R_2 \leq \left(1 - 4e^{-T}\right)\log(N) - 2(1 + e^{-T})h_2\left(\frac{e^{-T}}{1 + e^{-T}}\right)$$

for all $T > 0$, where $h_2(p)$ is the binary entropy function.

Maximal vs. Average Error

In network information theory, the channel capacity may depend on the error criterion.

- Maximal error: $P_e^{(n)} = \max_{m_1, m_2} \Pr(\text{error}|m_1, m_2)$
- Average error: $\overline{P}_e^{(n)} = \frac{1}{M_1 M_2} \sum_{m_1, m_2} \Pr(\text{error}|m_1, m_2)$

Maximal vs. Average Error

In network information theory, the channel capacity may depend on the error criterion.

- Maximal error: $P_e^{(n)} = \max_{m_1, m_2} \Pr(\text{error} | m_1, m_2)$
- Average error: $\overline{P}_e^{(n)} = \frac{1}{M_1 M_2} \sum_{m_1, m_2} \Pr(\text{error} | m_1, m_2)$

Let $\mathcal{C}$ and $\overline{\mathcal{C}}$ denote the corresponding capacity regions for the classical MAC $P_{Y|X_1,X_2}$.

In general, $\mathcal{C} \subseteq \overline{\mathcal{C}}$.

Maximal vs. Average Error (Cont.)

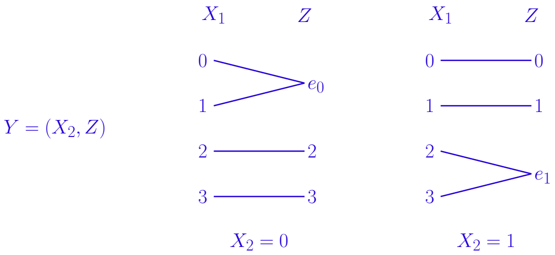
Without entanglement resources,

- In the single-user case (say, $\mathcal{X}_2 = \emptyset$), [Shannon 1948] [Wolfowitz 1957]

$$\mathcal{C} = \overline{\mathcal{C}}$$

- However, for some $P_{Y|X_1,X_2}$ [Dueck 1978]

$$\mathcal{C} \neq \overline{\mathcal{C}}$$



Maximal vs. Average Error (Cont.)

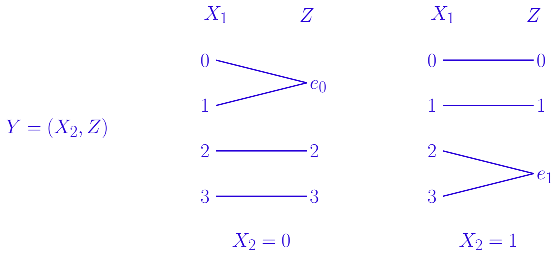
Without entanglement resources,

- In the single-user case (say, $\mathcal{X}_2 = \emptyset$),
  [Shannon 1948] [Wolfowitz 1957]

$$\mathcal{C} = \overline{\mathcal{C}}$$

- However, for some $P_{Y|X_1,X_2}$ [Dueck 1978]

$$\mathcal{C} \subsetneq \overline{\mathcal{C}}$$



Maximal vs. Average Error (Cont.)

Given entanglement resources, we observe that

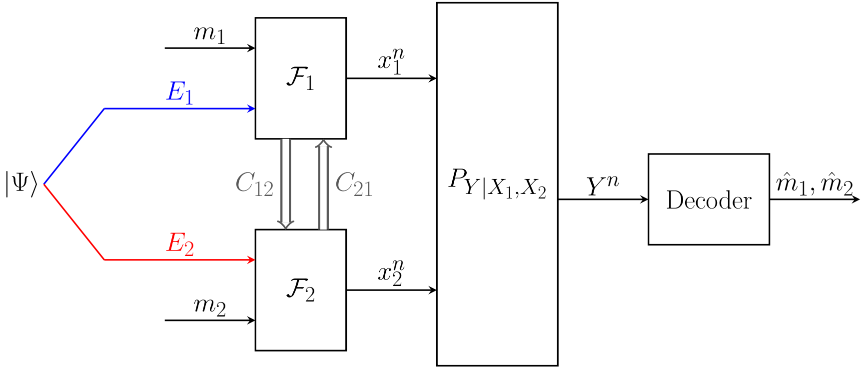
$$\mathcal{C}_{\text{ET}} = \overline{\mathcal{C}}_{\text{ET}}$$

Proof follows [Cai 2014] and resembles the robustification technique [Ahlswede 1986]:

- We use the entanglement to generate a shared random key at rate $R_{\text{key}} \approx 0$.
- The average over the key “replaces" the message average.

Entanglement and Conferencing

Suppose the senders can communicate with each other classically over rate-limited links ("bit-pipes") and share an entangled state $\Psi_{E_1E_2}$ a priori.



Entanglement and Conferencing (Cont.)

Observations: Classical Conferencing

- Both entanglement and common randomeness are static resources of *non-signaling correlation*, which cannot be used in order to send information.
- Conferencing is “stronger" in the sense that it is a *dynamic* resource of cooperation.
- Q: is conferencing at a low rate necessarily better than entanglement at a high rate?

  A: No

Observations: Classical Conferencing

- Both entanglement and common randomeness are static resources of *non-signaling correlation*, which cannot be used in order to send information.
- Conferencing is “stronger" in the sense that it is a *dynamic* resource of cooperation.
- Q: is conferencing at a low rate necessarily better than entanglement at a high rate?

  A: No

Entanglement and Conferencing (Cont.)

Observations: Classical Conferencing

- Both entanglement and common randomeness are static resources of *non-signaling correlation*, which cannot be used in order to send information.
- Conferencing is “stronger" in the sense that it is a *dynamic* resource of cooperation.
- Q: is conferencing at a low rate necessarily better than entanglement at a high rate?

  A: No

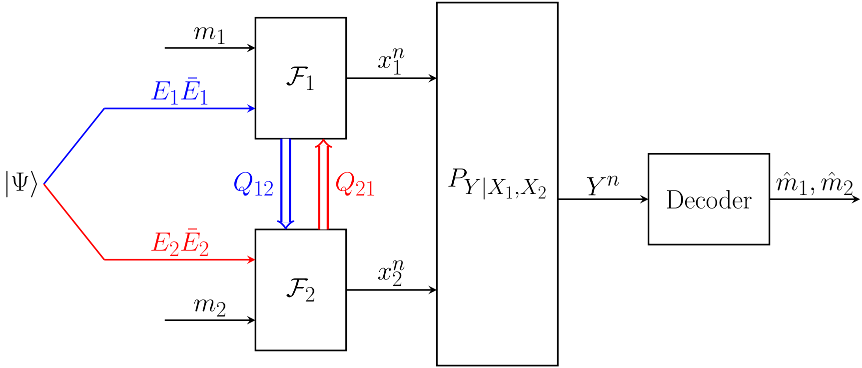
Entanglement and Conferencing (Cont.)

Observations: Classical Conferencing

- Both entanglement and common randomeness are static resources of *non-signaling correlation*, which cannot be used in order to send information.
- Conferencing is “stronger" in the sense that it is a *dynamic* resource of cooperation.
- Q: is conferencing at a low rate necessarily better than entanglement at a high rate?

  A: No

Now, suppose the senders can communicate over qubit-pipes and share an entangled state $\Psi_{E_1 E_2}$ a priori.



Entanglement and Conferencing (Cont.)

Observation: Quantum Conferencing

Each encoder can use superdense coding in order to **double** her conferencing rate.

Conclusion

We have considered communication over a two-user classical multiple-access channel (MAC) with entanglement resources shared between the transmitters before communication begins.

- capacity region for the *general* MAC with entangled transmitters.
- bounded auxiliary variables,
  impossible to bound dimensions for quantum ancillas (Tsirelson problem, MIP*=RE)

Conclusion (Cont.)

- Previous examples are a special case:
  - magic square: strictly higher than classical
  - linear equations: achievability requires infinite entanglement
- As opposed to the classical case [Dueck 1978], the capacity region with entangled transmitters is the same, whether it is a message-average or a maximal error criterion.

Conclusion (Cont.)

- Previous examples are a special case:
  - magic square: strictly higher than classical
  - linear equations: achievability requires infinite entanglement
- As opposed to the classical case [Dueck 1978], the capacity region with entangled transmitters is the same, whether it is a message-average or a maximal error criterion.
- Entanglement can increase the conferencing rate over qubit-pipe links

Thank You

Maximal Error Analysis

Given entanglement resources, we observe that

$$\mathcal{C}_{\mathsf{ET}} = \overline{\mathcal{C}}_{\mathsf{ET}}$$

Proof outline: Suppose we have a code with $\overline{P}_e^{(n)} \leq \lambda$.
Consider the semi-average error

$$\overline{Q}_e(m_2) = \frac{1}{M_1} \sum_{m_1} \Pr(\mathsf{error}|m_1, m_2)$$

Maximal Error Analysis

Given entanglement resources, we observe that

$$\mathcal{C}_{\mathsf{ET}} = \overline{\mathcal{C}}_{\mathsf{ET}}$$

Proof outline: Suppose we have a code with $\overline{P}_e^{(n)} \leq \lambda$.
Consider the semi-average error

$$\overline{Q}_e(m_2) = \frac{1}{M_1} \sum_{m_1} \Pr(\text{error}|m_1, m_2)$$

Throw away the worst half of $\{1, \ldots, M_2\}$.

Maximal Error Analysis (Cont.)

Since the average of $\overline{Q}_e(m_2)$ over the original set is $P_e^{(n)} \leq \lambda$, we have

$$\overline{Q}_e(m_2) \leq 2\lambda$$

for all messages in the remaining set $\mathcal{M}_2'$.

Maximal Error Analysis (Cont.)

Since the average of $\overline{Q}_e(m_2)$ over the original set is $P_e^{(n)} \leq \lambda$, we have

$$\overline{Q}_e(m_2) \leq 2\lambda$$

for all messages in the remaining set $\mathcal{M}_2'$.

Let Alice 1 draw a uniformly distributed key $L \in \{1, \ldots, n^2\}$. She can send this key to Bob using a code of length $o(n)$ with a 2λ-error.

Maximal Error Analysis (Cont.)

Alice 1 uses a sequence of permutations $\pi_1, \ldots, \pi_{n^2}$ over the message set $\{1, \ldots, M_1\}$. Given a key $L = \ell$, she encodes using $\mathcal{F}^{\pi_\ell(m_1)}$.

For a uniformly distributed permutation π, we have $\Pr(\pi(m_1) = m_1') = \frac{(M_1-1)!}{M_1!} = \frac{1}{M_1}$. Thus,

$$\begin{aligned}
\mathbb{E}[\Pr(\text{error}|\pi(m_1), m_2)] &= \sum_{m_1'} \Pr\big(\pi(m_1) = m_1'\big) \cdot \Pr\big(\text{error}|m_1', m_2\big) \\
&= \frac{1}{M_1} \sum_{m_1'} \Pr\big(\text{error}|m_1', m_2\big) \\
&= \overline{Q}_e^{(n)}(m_2) \\
&\leq 2\lambda \qquad\qquad \forall m_2 \in \mathcal{M}_2'
\end{aligned}$$

Maximal Error Analysis (Cont.)

Then, based on the Chernoff bound, for an i.i.d. sequence of uniform permutations $\pi_1, \ldots, \pi_{n^2}$,

$$\Pr\left(\frac{1}{n^2}\sum_{\ell=1}^{n^2} \Pr(\text{error}|\pi_\ell(m_1), m_2) > 7\lambda\right) \leq e^{-\lambda n^2}$$

Therefore, there exists a realization $\pi_1, \ldots, \pi_{n^2}$ such that

$$\frac{1}{n^2}\sum_{\ell=1}^{n^2} \Pr(\text{error}|\pi_\ell(m_1), m_2) \leq 7\lambda \qquad \forall m_1, m_2 \in \mathcal{M}_2' \qquad \square$$